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NOTE ON THE INTERNAL PRESSURES AT ANY POINT WITHIN A BODY AT REST.

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One of the theorems respecting the internal pressures at any point within a body at rest, to which Mr. Rankine alluded in this *Journal* (No. xxv. p. 54), is so elegant that it seems to deserve a place in our University mathematics.

The following proof of it differs slightly from that given by M. Lamé, in his *Leçons sur la Théorie Mathématique de l'élasticité des corps solides*, and the tensions or internal pressures to which it refers are only restricted to the meaning given by the annexed definition.

Def. Suppose a body to be at rest under the action of any external forces, its constituent particles being maintained in their positions of equilibrium by their mutual actions; then if the portion of the body cut off by any imaginary plane be removed, the equilibrium of the particles in the remaining portion may evidently still be preserved undisturbed by applying to each indefinitely small element ϖ of the plane surface, so formed, a force ϖE having the requisite magnitude and direction: designating the position of ϖ by the letter M, the force E is taken to be the measure per unit of surface of the internal tension of the body at the point M exerted upon the plane area ϖ.

This tension E will generally vary at the same point M for different positions of ϖ, and the theorem in question asserts that if a line be drawn from M to represent it in magnitude and direction, the locus of its extremity will be an ellipsoid; and further, that the plane of ϖ is diametral to those chords of another given surface of the second order, having M for its centre, which are parallel to the direction of E.

Although Mr. Rankine in the before-mentioned paper has enunciated the two fundamental theorems upon which all investigations in this subject are based, and has sketched out the nature of their proof, it will be convenient to re-state them here: they are,

(α) If three elementary planes be taken at right angles to each other through the same point within any body, and if $X_1 Y_1 Z_1$, $X_2 Y_2 Z_2$, $X_3 Y_3 Z_3$ represent the resolved parts of the tensions upon them, estimated according to the preceding definition, in directions parallel to their lines of intersection taken as coordinate axes of x, y, and z respectively, six only of these nine resolved forces are independent; for, the first plane being that perpendicular to the axis of x, the second that perpendicular to the axis of y, and similarly for z; then $Y_3 = Z_2$, $Z_1 = X_3$, $X_2 = Y_1$.

Of these forces the three which are normal to their respective planes may very well be designated by the symbols N_1, N_2, N_3, while the three others, as being tangential to their respective planes, may be distinguished by the letters T_1, T_2, T_3; so that the resolved tensions will be

N_1, T_3, T_2 for the plane perpendicular to axis of x,
T_3, N_2, T_1 . . . . . y,
T_2, T_1, N_3 . . . . . z.

(β) If E, E' represent the tensions called into action upon two elementary planes ϖ, ϖ', respectively, situated at the same point of a solid body; and if L, L' be the respective normals to these planes; then the projection of E upon L' is always equal to the projection of E' upon L.

Assuming these two theorems, take at any point O of a solid body three rectangular axes, and let, in accordance with (α), N_1, N_2, N_3, T_1, T_2, T_3 represent the resolved parts of the tensions, which are exerted at the point O upon three very small planes respectively perpendicular to the coordinate axes: also let ϖ be any other small plane at O, its direction cosines being l, m, n; draw OP to represent both in magnitude and direction the tension upon ϖ; put ρ for its length and l', m', n' for its direction cosines. Then, from theorem (3), we get by the law of projections the following relation between the tension upon ϖ and that upon the plane which is perpendicular to the axis of x,

$$\left.\begin{aligned} \rho l' &= lN_1 + mT_3 + nT_2 \\ \text{similarly} \quad \rho m' &= lT_3 + mN_2 + nT_1 \\ \rho n' &= lT_2 + mT_1 + nN_3 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (A).$$

If it be possible that ρ should ever be perpendicular to ϖ, then would $l' = l$, $m' = m$, $n' = n$, and equations (A) would become

$$\left.\begin{aligned} (N_1 - \rho)\, l + T_3 m + T_2 n &= 0 \\ T_3 l + (N_2 - \rho)\, m + T_1 n &= 0 \\ T_2 l + T_1 m + (N_3 - \rho)\, n &= 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots (B),$$

which, together with

$$l^2 + m^2 + n^2 = 1 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

would form four equations for determining ρ and its direction cosines.

Eliminating l, m, and n from (B), we obtain

$$(N_1 - \rho)(N_2 - \rho)(N_3 - \rho) - (N_1 - \rho)\, T_1^2 - (N_2 - \rho)\, T_2^2 - (N_3 - \rho)\, T_3^2 + 2 T_1 T_2 T_3 = 0 \ldots\ldots (2),$$

a well-known cubic which necessarily gives three real roots to ρ; call these ρ_1, ρ_2, ρ_3; the corresponding values of l, m, and n may be easily found from equations (B) and (1).

We see then that there are three, and in general only three, planes passing through any point O of a solid, the tensions upon which are perpendicular to them; these planes, and consequently their tensions ρ_1, ρ_2, ρ_3, are at right angles to each other; for if we solve equations (A) with respect to l, m, and n, we get

$$\left.\begin{aligned} \frac{\lambda}{\rho}\, l &= A_1 l' + B_3 m' + B_2 n' \\ \frac{\lambda}{\rho}\, m &= B_3 l' + A_2 m' + B_1 n' \\ \frac{\lambda}{\rho}\, n &= B_2 l' + B_1 m' + A_3 n' \end{aligned}\right\} \ldots\ldots\ldots\ldots (C),$$

λ, A_1, A_2, A_3, B_1, B_2, B_3 being put for certain combinations of N and T which arise from the operation; and these equations shew that the plane of ϖ is diametral to the system of chords drawn parallel to ρ in the surface of the second order, whose equation is

$$A_1 x^2 + A_2 y^2 + A_3 z^2 + 2B_1 yz + 2B_2 zx + 2B_3 xy + K = 0 \ldots (3),$$

and it is the property of such a surface that the three planes, which bisect their chords at right angles, are perpendicular to each other.

These tensions ρ_1, ρ_2, ρ_3 may be very conveniently termed the principal tensions at O. Take their directions for new axes of coordinates, and let x, y, z be the coordinates of P when referred to them, and l_1, m_1, n_1 the direction-cosines of ϖ, then, by theorem (β), equating the projection of ρ upon the axis of x to that of ρ_1 upon the normal to ϖ,

$$\left.\begin{aligned} x &= l_1\rho_1 \\ y &= m_1\rho_2 \\ z &= n_1\rho_3 \end{aligned}\right\} \text{similarly} \quad \ldots\ldots\ldots\ldots\ldots\ldots (D);$$

$$\text{therefore} \qquad \frac{x^2}{\rho_1^2} + \frac{y^2}{\rho_2^2} + \frac{z^2}{\rho_3^2} = 1 \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which shews that ρ, the magnitude of the tension upon any elementary plane ϖ at O, is represented by the portion of OP cut off by the ellipsoid whose equation is (4) and whose semiaxes have the same magnitude and direction as $\rho_1\rho_2\rho_3$, which represent the principal tensions at O.

It has been already remarked that OP is always parallel to the systems of chords, which are bisected by the plane of ϖ, in that surface of the second order whose equation referred to the first set of axes was (3); ϖ is therefore parallel to the tangent plane at the point where OP meets this surface.

The direction-cosines of the new axes are given by the equations (B) and (1): but the equations (B) are identical with the following, which result from putting $l' = l$, $m' = m$, $n' = n$, in (C),

$$\left.\begin{aligned} \left(A_1 - \frac{\lambda}{\rho}\right) l + B_3 m + B_2 n &= 0 \\ B_3 l + \left(A_2 - \frac{\lambda}{\rho}\right) m + B_1 n &= 0 \\ B_2 l + B_1 m + \left(A_3 - \frac{\lambda}{\rho}\right) n &= 0 \end{aligned}\right\} \ldots\ldots\ldots (E);$$

and therefore

$$\left(A_1 - \frac{\lambda}{\rho}\right)\left(A_2 - \frac{\lambda}{\rho}\right)\left(A_3 - \frac{\lambda}{\rho}\right)$$

$$- \left(A_1 - \frac{\lambda}{\rho}\right) B_1^2 - \left(A_2 - \frac{\lambda}{\rho}\right) B_2^2 - \left(A_3 - \frac{\lambda}{\rho}\right) B_3^2 + 2B_1B_2B_3 = 0 \ldots (5)$$

must be identical with (2) and give the same values to ρ: but the transformation of the equation (3) to axes of co-

ordinates whose direction cosines are given by (E), produces, as is well known, an equation not involving the products of the coordinates and one where the coefficients of x^2, y^2, z^2 are the roots of (5); therefore (3), by the alteration of co-ordinate axes, becomes

$$\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\frac{z^2}{\rho_3}+\frac{K}{\lambda}=0 \ldots\ldots\ldots\ldots\ldots (6).$$

If $\rho_1\rho_2\rho_3$, the roots of (2), be all of the same sign, *i.e.* if the principal tensions at O be all, either compressive or tensile, (6) must be an ellipsoid, and therefore ρ always, whatever be the position of ϖ, represents a force of the same kind as ρ_1, ρ_2, ρ_3, whether compressive or tensile.

If however the roots of (3) be not all of the same sign, then K in (6) may be either positive or negative, and therefore (6) must be taken to represent at once two hyperboloids, the one of two, the other of one sheet, both having a common asymptotic cone whose equation is

$$\frac{x^2}{\rho_1}+\frac{y^2}{\rho_2}+\frac{z^2}{\rho_3}=0 \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

In this case when OP meets the hyperboloid of two sheets, the force represented by it must be of the same kind as those represented by the two *like* axes of the surface, and when it meets the other hyperboloid the force must be the same as that represented by the single *unlike* axis: in intermediate positions to these OP must lie upon the cone (7), and be therefore in the plane of ϖ, which is always parallel to the tangent-plane at the point where OP meets (6).

If one of the roots of (2), as ρ_3, $=0$, then there is one plane through O upon which there is no tension, and consequently, by theorem (β), the directions of the tensions upon all other planes through O must lie in this plane; hence x, y, z being as before the coordinates of P referred to the directions of the principal tension as axes, and l_1, m_1, n_1 the direction-cosines of ϖ, $z=0$: but $\frac{z}{\rho_3}=n_1=\cos\gamma$, if γ represent the inclination of ϖ to the plane of xy; therefore

$$\frac{x^2}{\rho_1^2}+\frac{y^2}{\rho_2^2}+\cos^2\gamma=1,$$

or

$$\frac{x^2}{\rho_1^2}+\frac{y^2}{\rho_2^2}=\sin^2\gamma \ldots\ldots\ldots\ldots\ldots (8),$$

the equation to an ellipse in the plane of xy, representing the locus of the extremity of OP for all such planes ϖ as are inclined at an angle γ to plane of xy.

Also the direction-cosines of the line in which ϖ intersects the plane of xy are $l_1 m_1$, *i.e.* (from (D)) $\frac{x}{\rho_1}$, $\frac{y}{\rho_2}$, and they are therefore proportional to $\frac{x_1}{\rho_1}$ and $\frac{y_1}{\rho_2}$ respectively, where $x_1 y_1$ are the coordinates of the point where OP produced meets the conic section whose equation is

$$\frac{x^2}{\rho_1} + \frac{y^2}{\rho_2} = K' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Hence this line of intersection is parallel to the tangent to (9) at point $x_1 y_1$, and the relation between OP and ϖ is completely determined.

If ρ_1 and ρ_2 have the same signs, (9) is an ellipse, but if opposite signs it is an hyperbola and its conjugate, having the same asymptotes, because K may be either positive or negative; the nature of the pressure represented by OP may be obtained by reasoning similar to that just employed in regard to the hyperboloid (6).

If two of the roots of (2), as ρ_3 and ρ_2, each = 0, then the theorem (β) shews that the tension upon every small plane through O is in the same straight line as the remaining principal tension ρ_1; its value for any particular plane ϖ is by the same theorem equal to the projection of ρ_1 upon the normal to ϖ.

If two of the roots of equation (2) be equal, (4) and (6) represent surfaces of revolution, and it will therefore follow that all semidiameters of (4), which lie in a plane perpendicular to the axis of revolution, represent tensions which are normal to the planes upon which they are exerted. If the three roots of (2) be equal, then both (4) and (6) are spheres; all the tensions are therefore normal tensions and all have the same magnitude.

Clare Hall, Cambridge,
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SOLUTIONS OF PROBLEMS.

(Prob. 2, vol. VIII. p. 188).

If from a point in the circumference of a vertical circle two heavy particles be successively projected along the curve, their initial velocities being equal and either in the same or in opposite directions, the subsequent motion will be such that a straight line joining the particles at any instant will touch a given circle.

Note. The particles are supposed not to interfere with each other's motion.

THE direct analytical proof would involve the properties of elliptic integrals, but it may be made to depend upon the following geometrical theorems.

(1) If from a point in one of two circles a right line be drawn cutting the other, the rectangle contained by the segments so formed is double of the rectangle contained by a line drawn from the point perpendicular to the *radical axis* of the two circles, and the line joining their centres.

The radical axis is the line joining the points of intersection of the two circles. It is always a real line, whether the points of intersection of the circles be real or imaginary, and it has the geometrical property—that if from any point on the radical axis, straight lines be drawn cutting the circles, the rectangle contained by the segments formed by one of the circles is equal to the rectangle contained by the segments formed by the other.

The analytical proof of these propositions is very simple, and may be resorted to if a geometrical proof does not suggest itself as soon as the requisite figure is constructed.

If A, B be the centres of the circles, P the given point in the circle whose centre is A, a line drawn from P cuts the first circle in p, the second in Q and q, and the radical axis in R. If PH be drawn perpendicular to the radical axis, then

$$PQ.Pq = 2AB.HP.$$

COR. If the line be drawn from P to *touch* the circle in T, instead of cutting it in Q and q, then the square of the tangent PT is equal to the rectangle $2AB.HP$.

Similarly, if ph be drawn from p perpendicular to the radical axis

$$pT^2 = 2AB.hp.$$

Hence, if a line be drawn touching one circle in T, and cutting the other in P and p, then

$$(PT)^2 : (pT)^2 :: HP : hp.$$

(2) If two straight lines touching one circle and cutting another be made to approach each other indefinitely, the small arcs intercepted by their intersections with the second circle will be ultimately proportional to their distances from the point of contact.

This result may easily be deduced from the properties of the similar triangles $P'PT$ and $p'pT$.

Cor. If particles P, p be constrained to move in the circle A, while the line Pp joining them continually touches the circle B, then the velocity of P at any instant is to that of p as PT to pT; and conversely, if the velocity of P at any instance be to that of p as PT to pT, then the line Pp will continue to be a tangent to the circle B.

Now let the plane of the circles be vertical and the radical axis horizontal, and let gravity act on the particles P, p. The particles were projected from the same point with the same velocity. Let this velocity be that due to the depth of the point of projection below the radical axis. Then the square of the velocity at any other point will be proportional to the perpendicular from that point on the radical axis; or, by the corollary to (1), if P and p be at any time at the extremities of the line PTp, the square of the velocity of P will be to the square of the velocity of p as PH to ph, that is, as $(PT)^2$ to $(pT)^2$. Hence, the velocities of P and p are in the proportion of PT to pT, and therefore, by the corollary to (2), the line joining them will continue a tangent to the circle B during each instant, and will therefore remain a tangent during the motion.

The circle A, the radical axis, and one position of the line Pp, are given by the circumstances of projection of P and p. From these data it is easy to determine the circle B by a geometrical construction.

It is evident that the character of the motion will determine the position of the circle B. If the motion is oscillatory, B will intersect A. If P and p make complete revolutions in the same direction, B will lie entirely within A, but if they move in opposite directions, B will lie entirely above the radical axis.

If any number of such particles be projected from the same point at equal intervals of time with the same direction and velocity, the lines joining successive particles at any instant will be tangents to the same circle; and if the time of a complete revolution, or oscillation, contain n of these intervals, then these lines will form a polygon of n sides,

and as this is true at any instant, any number of such polygons may be formed.

Hence, the following geometrical theorem is true:

"If two circles be such that n lines can be drawn touching one of them and having their successive intersections, including that of the last and first, on the circumference of the other, the construction of such a system of lines will be possible, at whatever point of the first circle we draw the first tangent."

(Prob. 3, vol. VIII. p. 188).

A transparent medium is such that the path of a ray of light within it is a given circle, the index of refraction being a function of the distance from a given point in the plane of the circle.

Find the form of this function and shew that for light of the same refrangibility—

(1) The path of *every ray within the medium* is a circle.

(2) All the rays proceeding from any point in the medium will meet accurately in another point.

(3) If rays diverge from a point without the medium and enter it through a spherical surface having that point for its centre, they will be made to converge accurately to a point within the medium.

LEMMA I. Let a transparent medium be so constituted, that the refractive index is the same at the same distance from a fixed point, then the path of any ray of light within the medium will be in one plane, and the perpendicular from the fixed point on the tangent to the path of the ray at any point will vary inversely as the refractive index of the medium at that point.

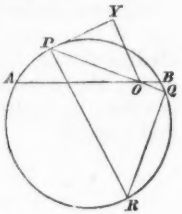
We may easily prove that when a ray of light passes through a spherical surface, separating a medium whose refractive index is μ_1 from another where it is μ_2, the plane of incidence and refraction passes through the centre of the sphere, and the perpendiculars on the direction of the ray before and after refraction are in the ratio of μ_2 to μ_1. Since this is true of any number of spherical shells of different refractive powers, it is also true when the index of refraction varies continuously from one shell to another, and therefore the proposition is true.

LEMMA II. If from any fixed point in the plane of a circle, a perpendicular be drawn to a tangent at any point of the circumference, the rectangle contained by this perpendicular and the diameter of the circle is equal to the square of the line joining the point of contact with the fixed

point, together with the rectangle contained by the segments of any chord through the fixed point.

Let APB be the circle, O the fixed point; then

$$OY.PR = OP^2 + AO.OB.$$



Produce PO to Q, and join QR, then the triangles OYP, PQR are similar; therefore

$$OY.PR = OP.PQ$$
$$= OP^2 + OP.OQ;$$
$$\therefore \quad OY.PR = OP^2 + AO.OB.$$

If we put in this expression $AO.OB = a^2$,

$$PO = r, \quad OY = p, \quad PR = 2\rho,$$

it becomes

$$2p\rho = r^2 + a^2,$$
$$p\rho = \frac{r^2 + a^2}{2\rho}.$$

To find the law of the index of refraction of the medium, so that a ray from A may describe the circle APB, μ must be made to vary inversely as p by Lemma I.

Let $AO = r_1$, and let the refractive index at $A = \mu_1$; then generally

$$\mu = \frac{C}{p} = \frac{2C\rho}{a^2 + r^2};$$

but at A

$$\mu_1 = \frac{2C\rho}{a^2 + r_1^2},$$

therefore

$$\mu = \mu_1 \frac{a^2 + r_1^2}{a^2 + r^2}.$$

The value of μ at any point is therefore independent of ρ, the radius of the given circle; so that the same law of refractive index will cause any other ray to describe another circle, for which the value of a^2 is the same. The value of OB is $\frac{a^2}{r_1}$, which is also independent of ρ; so that every ray which proceeds from A must pass through B.

Again, if we assume μ_0 as the value of μ when $r = 0$,

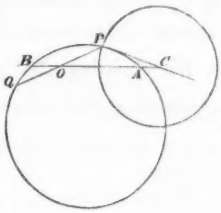
$$\mu_0 = \mu_1 \frac{a^2 + r_1^2}{a^2};$$

therefore

$$\mu = \mu_0 \frac{a^2}{a^2 + r^2},$$

a result independent of r_1. This shews that any point A' may be taken as the origin of the ray instead of A, and that the path of the ray will still be circular, and will pass through another point B' on the other side of O, such that $OB' = \frac{a^2}{OA'}$.

Next, let CP be a ray from C, a point without the medium, falling at P on a spherical surface whose centre is C.



Let O be the fixed point in the medium as before. Join PO, and produce to Q till $OQ = \frac{a^2}{OP}$. Through Q draw a circle touching CP in P, and cutting CO in A and B; then PBQ is the path of the ray within the medium.

Since CP touches the circle, we have

$$CP^2 = CA.CB,$$
$$= (CO - OA)(CO + OB);$$

but $$OA = \frac{a^2}{OB};$$

therefore $$CP^2 = CO^2 + CO\left(OB - \frac{a^2}{OB}\right) - a^2,$$

an equation whence OB may be found, B being the point in the medium through which all rays from C pass.

Note. The possibility of the existence of a medium of this kind possessing remarkable optical properties, was suggested by the contemplation of the structure of the crystalline lens in fish; and the method of searching for these properties was deduced by analogy from Newton's *Principia*, lib. I. prop. VII.

It would require a more accurate investigation into the law of the refractive index of the different coats of the lens to test its agreement with the supposed medium, which is an optical instrument theoretically perfect for homogeneous light, and might be made achromatic by proper adaptation of the dispersive power of each coat.

On the other hand, we find that the law of the index of refraction which would give a minimum of aberration for a sphere of this kind placed in water, gives results not discordant with facts, so far as they can be readily ascertained.

(Prob. 4, vol. VIII. p. 188.)

A series of waves, which at sea are twenty feet long from crest to crest, and three feet high from hollow to crest, break on a shore which is parallel to their breadth. How much heat is developed per hour on each foot of the shore, and how much would the temperature of 180 cubic feet of fresh water be raised by receiving an equal quantity? [The form of a wave at sea, of which the height is a small fraction of l, its length, is approximately the curve of sines; its velocity of propagation is $\sqrt{\frac{gl}{2\pi}}$; and its mechanical energy is half that of a double elevation and depression of the same form without velocity.]

By choosing the most convenient origin and axes, the equation of the wave will be of the form $y = k \sin \frac{x}{a}$; where k and a are constants depending on the height and length of the wave.

When $y = 0$, $x_0 = am\pi$, and as between two successive crests the wave meets the axis of x twice, we have

$$l = a(m\pi) - a(m-2)\,\pi = 2a\pi.$$

Also when $x = a(m + \frac{1}{2})\,\pi$, $y = k$, a maximum;

and when $x = a(m + \frac{3}{2})\,\pi$, $y = -k$, a minimum;

so that denoting the height of the wave from hollow to crest by h, $k = \frac{1}{2}h$, and the equation of the wave becomes

$$y = \tfrac{1}{2}h \sin \frac{2\pi x}{l}.$$

The proof that its mechanical energy is half that of a double elevation and depression of the same form, without velocity, is not difficult, but as it is granted in the enunciation we need not give it. By means of this proposition however we obtain a very important simplification, as in the standing wave the motion of any particle is only in a vertical straight line. The equation of the standing wave will of course be

$$y = h \sin \frac{2\pi x}{l}.$$

It is evident that the mechanical energy of a wave is the amount of work which we can obtain by bringing it to a level, without producing currents of any kind. To estimate this in the case of a standing wave, suppose the wave made rigid, and that the portion above the horizontal plane containing the axis of x be turned over into the depression below that plane. Let P be the weight of a slice of the wave above the axis of x, one foot in breadth, and y_1 the ordinate of its centre of gravity, then $2Py_1$ is the value

of the work that can be obtained by this inversion, in other words is the mechanical energy of a standing wave.

Denoting this by $2W$, we have

$$2W = 2Py_1 = 2\int_0^{\frac{1}{2}l} dx \int_0^y py\,dy,$$

and as 62·447 pounds is the weight of a cubic foot of distilled water, and 1·027 is the specific gravity of sea water,

$$p = 62{\cdot}447 \times 1{\cdot}027,$$

$$2W = p\int_0^{\frac{1}{2}l} y^2 dx = ph^2 \int_0^{\frac{1}{2}l} \left(\sin\frac{2\pi x}{l}\right)^2 dx,$$

$$= ph^2 \left\{ -\frac{l}{2\pi}\sin\frac{2\pi x}{l}\cos\frac{2\pi x}{l} + \int\left(\cos\frac{2\pi x}{l}\right)^2 dx \right\}_0^{\frac{1}{2}l},$$

$$= ph^2 \left\{ -\frac{l}{4\pi}\sin\frac{2\pi x}{l}\cos\frac{2\pi x}{l} + \frac{x}{2} + C \right\}_0^{\frac{1}{2}l},$$

$$= \tfrac{1}{4}ph^2 l \text{ in foot pounds.}$$

Mechanical energy of one *moving* wave $= W = \frac{1}{8}ph^2 l$. Thermal equivalent of the energy of one moving wave $= \frac{1}{8}\frac{ph^2 l}{J}$, J being the number of foot pounds of work equivalent to one unit of heat. Again, velocity of wave $= \sqrt{\frac{gl}{2\pi}}$. Period $= \frac{l}{v}$. Number of waves per hour $= \frac{1}{l}\sqrt{\frac{gl}{2\pi}} \times 60^2$. Hence if units of heat developed per hour opposite one foot of the shore $= H$,

$$H = \frac{1}{8}\frac{ph^2 l}{J} \cdot \frac{1}{l}\sqrt{\frac{gl}{2\pi}} \times 60^2,$$

$$H = \sqrt{\frac{gl}{2\pi}} \cdot \frac{60^2 h^2 p}{8J}$$

If a quantity equal to this be distributed among 180 cubic feet of fresh water, or $180 \times 62{\cdot}447$ pounds, each pound will obtain

$$\sqrt{\frac{gl}{2\pi}} \cdot \frac{60^2 h^2 p}{8J \times 180 \times 62{\cdot}447} = \sqrt{\frac{gl}{2\pi}} \cdot \frac{60^2 h^2 \times 1{\cdot}027}{8 \times J \times 180} \text{ units;}$$

so that if T_0 be the original temperature of the fresh water, and T its temperature at the end of the hour,

$$T = T_0 + \sqrt{\frac{gl}{2\pi}} \cdot \frac{60^2 h^2 \times 1{\cdot}027}{8J \times 180}.$$

Ex. If $l = 20$ feet and $h = 3$ feet,

$$H = \sqrt{\left(\frac{32{\cdot}2 \times 40}{2 \times \pi}\right)} \cdot \frac{60^2 \times 3^3 \times 62{\cdot}447 \times 1{\cdot}027}{8 \times J}.$$

From the experiments of Mr. Joule, J is found to be about 772, if we employ the Fahrenheit unit of heat, and consequently about 1390 for the Centigrade. Employing these numbers successively in the formula, we have

$$H = 3406{\cdot}22 \text{ units Faht.}$$
$$= 1891{\cdot}80 \text{ units Cent.}$$

From the other formula we find

$$T = T_0 + {\cdot}3030^\circ \text{ Faht.}$$
$$T = T_0 + {\cdot}1683^\circ \text{ Cent.}$$

(Prob. 6, vol. VIII. p. 189.)

Find the amount of "potential energy" (mechanical effect of such a kind as that of weights raised) that can be obtained by equalizing the temperature of two bodies given at different uniform temperatures, and determine the common temperature to which they are reduced.

Ex. 1. Let the bodies be of equal constant thermal capacities, and let their temperatures be 0° and 100° respectively.

Ex. 2. Let the bodies be masses W and W' of water, and let the temperatures at which they are given be 15° and 20° respectively.

For convenience call the two bodies A and B, A being higher in temperature than B. Let T be the initial temperature of the hotter body A, T_1 the initial temperature of B, and θ the common temperature to which they are reduced. The thermal capacities will depend on the temperatures of A and B, and will be expressible in some function of their temperatures.

Let the initial thermal capacity of $A = \phi' T$, of $B = \psi' T_1$. Generally, thermal capacity of $A = \phi' t$, of $B = \psi' t$, $\phi' t$ and $\psi' t$ being the differential coefficients, with respect to t, of some functions ϕt and ψt.

The simplest method in our estimation of solving this problem, is to suppose a third body C of temperature θ, and instead of employing one engine having A for source and B for refrigerator, employ two engines, one having A for source and C for refrigerator, the other having C for source and B for refrigerator.

As we shall employ perfect thermodynamic engines, it is easy to shew that if we take out as much heat from C

considered as a source as we give to it as a refrigerator, we shall obtain exactly as much mechanical effect by employing both engines, as by employing one with source A and refrigerator B. In short if the above relation were not true, then suppose the two engines produced more mechanical effect than the one, the two could then be made to drive the one backwards, and raise weights in addition, and we should be driven to the absurdity of deriving mechanical effect by drawing heat from the refrigerator.

(1) Consider then an engine drawing heat from A at a variable temperature, and giving part of it out to C at temperature θ, and obtaining as much work as possible by the transference.

To diminish the temperature of A from t to $t-dt$, $\phi' t\,dt$ units must be taken away, hence the

$$\text{whole heat given out by } A = \int_{\theta}^{T} \phi' t\,dt = (\phi T - \phi\theta).$$

Mechanical effect gained by the fall in temperature of A from t to $t-dt$

$$= J\phi' t\,dt\,\frac{t-\theta}{t+\frac{1}{E}}.$$

$\phi' t\,dt$ corresponding to H in the formula given in the last number of this *Journal* by Professor Thomson.

Total mechanical effect gained by allowing the temperature of A to fall from T to $\theta = P_1$ suppose, then we have

$$P_1 = J\int_{\theta}^{T} \phi' t\,dt\,\frac{t-\theta}{t+\frac{1}{E}}.$$

Why it should be $\int_{\theta}^{T}$ and not $\int_{T}^{\theta}$ needs no remark, if attention be paid to the signs.

$$P_1 = J\int_{\theta}^{T} \phi' t\,dt - J\int_{\theta}^{T} \phi' t\,dt\,\frac{\theta+\frac{1}{E}}{t+\frac{1}{E}}$$

$$= J\{\phi T - \phi\theta\} - J\left(\theta+\frac{1}{E}\right)(\phi_1 T - \phi_1\theta),$$

by supposing $$\int\frac{\phi' t\,dt}{t+\frac{1}{E}} = \phi_1 t + C;$$

but we found before $(\phi T - \phi\theta)$ is the number of units given out by A, and therefore the number of units given out to C must be

$$\left(\theta + \frac{1}{E}\right)(\phi_1 T - \phi_1\theta).$$

(2) Consider now an engine having C at constant temperature θ as source, and B at variable temperature as refrigerator. Consider the circumstances when the temperature of B is t. To raise the temperature of B from t to $t + dt$, $\psi' t\, dt$ units must be added.

Whole heat received by $B = \int_{T_1}^{\theta} \psi' t\, dt = (\psi\theta - \psi T_1)$.

As in raising B from t to $t + dt$, $\psi' t\, dt$ units are given to it, (the refrigerator) $\psi' t\, dt \dfrac{\theta + \frac{1}{E}}{t + \frac{1}{E}}$ units must have been supplied by the source C. Substituting this for H in the formula, we have

Mechanical effect gained in raising the temperature of B from t to $t + dt$

$$= J\psi' t\, dt \frac{\theta + \frac{1}{E}}{t + \frac{1}{E}} \cdot \frac{\theta - t}{\theta + \frac{1}{E}}$$

$$= J\psi' t\, dt \frac{\theta - t}{t + \frac{1}{E}}$$

Total mechanical effect that can be gained by raising B from T_1 to $\theta = P_2$ suppose, then

$$P_2 = J\int_{T_1}^{\theta} \psi' t\, dt \frac{\theta - t}{t + \frac{1}{E}},$$

$$P_2 = J\left(\theta + \frac{1}{E}\right)(\psi_1\theta - \psi_1 T_1) - J(\psi\theta - \psi T_1),$$

supposing $$\int \frac{\psi' t\, dt}{t + \frac{1}{E}} = \psi_1 t + C'.$$

But as the whole heat given out to B is, as we saw,

$$= (\psi\theta - \psi T_1),$$

the whole heat given out by C must have been

$$\left(\theta + \frac{1}{E}\right)(\psi_1\theta - \psi_1 T_1).$$

Now we have supposed the body C of such a nature, that its temperature is constant during the whole operation; but whether we suppose that it receives first all the heat it gets from A, and then gives off to B, or that both engines work together, and that the one takes out as fast as the other gives in, we have at least this condition, that *the amount of heat C gives out is exactly equal to the amount it receives*, for on no other supposition could we substitute two engines for one. This gives us the means of determining θ, for putting the quantity given by the first engine equal to that taken away by the second, we have

$$\left(\theta + \frac{1}{E}\right)(\phi_1 T - \phi_1\theta) = \left(\theta + \frac{1}{E}\right)(\psi_1\theta - \psi_1 T_1),$$

or

$$\phi_1\theta + \psi_1\theta = \phi_1 T + \psi_1 T_1,$$

from which θ may be obtained in any case.

Hence also we conclude that the terms multiplied by $\left(\theta + \frac{1}{E}\right)$ in the expressions for P_1 and P_2, given above, disappear in the addition, and we have the following expression for the required amount of "potential energy,"

$$P = P_1 + P_2 = J\{(\phi T - \phi\theta) - (\psi\theta - \psi T_1)\}.$$

Ex. 1. Let the thermal capacities be equal and constant.

$$\phi' t = \psi' t = C;$$

then

$$\phi T - \phi\theta = C(T - \theta),$$

and

$$\phi_1 T - \phi_1\theta = C \log \frac{T + \frac{1}{E}}{\theta + \frac{1}{E}},$$

$$\psi\theta - \psi T_1 = C(\theta - T_1),$$

$$\psi_1\theta - \psi_1 T_1 = C \log \frac{\theta + \frac{1}{E}}{T_1 + \frac{1}{E}}.$$

Substituting, we have

$$P = JC(T+T_1-2\theta),$$

and
$$\log\frac{T+\frac{1}{E}}{\theta+\frac{1}{E}} = \log\frac{\theta+\frac{1}{E}}{T_1+\frac{1}{E}};$$

whence
$$\left(T+\frac{1}{E}\right)\left(T_1+\frac{1}{E}\right) = \left(\theta+\frac{1}{E}\right)^2.$$

which gives $\theta = -\frac{1}{E} \pm \sqrt{\left\{\left(T+\frac{1}{E}\right)\left(T_1+\frac{1}{E}\right)\right\}}$,

$$P = JC\left[T+T_1+\frac{2}{E}-2\sqrt{\left\{\left(T+\frac{1}{E}\right)\left(T_1+\frac{1}{E}\right)\right\}}\right],$$

$$= JC\left\{\sqrt{\left(T+\frac{1}{E}\right)}-\sqrt{\left(T_1+\frac{1}{E}\right)}\right\}^2.$$

DEF. If we have a body at t Cent., $t+\frac{1}{E}$ is called its absolute temperature.

From the preceding equations we deduce the curious result, that the absolute common temperature is a mean proportional between the absolute original temperatures of A and B. The formula for P is also interesting.

Let $T = 100°$ Cent. and $T_1 = 0°$ Cent.

$$\theta = -273{\cdot}22404 + 319{\cdot}3',$$

$$= 46{\cdot}1093.$$

$$P = JC(100-92{\cdot}2186),$$

$$= JC \times 7{\cdot}7814.$$

Ex. (2). $A = W$ pounds, and $B = W'$ pounds of water.

Then, as the specific heat of water is nearly constant, and also is the standard unit, we have

$$\phi' t = W, \quad \psi' t = W',$$

$$\phi T - \phi\theta = W(T-\theta), \qquad \psi\theta - \psi T_1 = W'(\theta - T_1),$$

$$\phi_1 T - \phi_1\theta = W\log\frac{T+\frac{1}{E}}{\theta+\frac{1}{E}}, \quad \psi_1\theta - \psi_1 T_1 = W'\log\frac{\theta+\frac{1}{E}}{T+\frac{1}{E}}.$$

$$P = J\{W(T-\theta) - W'(\theta - T_1)\},$$
$$= J\{WT + W'T_1 - (W+W')\theta\}.$$

And
$$W \log \frac{T+\frac{1}{E}}{\theta+\frac{1}{E}} = W' \log \frac{\theta+\frac{1}{E}}{T_1+\frac{1}{E}},$$

$$\left(T+\frac{1}{E}\right)^{W}\left(T_1+\frac{1}{E}\right)^{W'} = \left(\theta+\frac{1}{E}\right)^{W+W'}.$$

Let $T = 20$, $T_1 = 15$, $W = 50$, $W' = 40$.

$$\theta = -273{\cdot}22404 + \{293{\cdot}22404^5 \times 288{\cdot}22404^4\}^{\frac{1}{9}},$$
$$= -273{\cdot}22404 + 290{\cdot}9912,$$
$$= 17{\cdot}7672.$$

$$P = 1390 \times {\cdot}952,$$
$$= 1323{\cdot}28.$$

C. A. S.

EXERCISES IN THE HYPERDETERMINANT CALCULUS.

By the Rev. George Salmon.

The readers of this *Journal* have been made acquainted, by several recently-published papers of Mr. Sylvester, with various theorems connecting together different derivatives of homogeneous functions of several variables. A method commonly employed in discovering such relations is the method of "Canonical Forms," in which, when it has been ascertained what is the simplest form to which the general expression is capable of being reduced, the relations found to be true between the derivatives of this simplest form are known to be true in general. Even however when applied to the equations of curves of the fourth degree the simplest expressions become so complicated that it is difficult to work with them, and at all events this method has never appeared to me thoroughly satisfactory, it rather shewing that a relation is true than making it plain why it is true. A recent perusal of Mr. Cayley's remarkable

paper on Linear Transformations (*Journal*, vol. I. p. 104), which I am ashamed to say I never studied carefully before, leads me to think that by the help of the notation there employed, much further progress in this theory may be expected to be made. His notation has the advantage of not only shewing that each derivative he deals with is an invariant or covariant as the case may be, but also of completely identifying each particular derivative and affording the means of comparing it with others. It is true that as the degree of the derivative increases, its symbolical expression becomes more and more complicated; still I should hope that if more attention were directed to this method, simpler means of manipulating these expressions would be discovered. In the following pages however no new principle is made use of; they contain merely selections from different examples to which I applied Mr. Cayley's method when I became acquainted with it, and which I publish in the hope that they may induce others to turn their attention to the same subject.

I commence with the case of homogeneous functions of two variables. In the paper just referred to Mr. Cayley proves that if in a function of equations between several systems of pairs of variables x_1, y_1, x_2, y_2, &c., the analogous variables be transformed in each case by the same linear substitutions; then (as is evident enough) the differentials $\frac{d}{dx_1}, \frac{d}{dy_1}, \frac{d}{dx_2}, \frac{d}{dy_2}$, &c., will also be transformed by linear substitutions; that consequently any symbol of operation of the form $\left(\frac{d}{dx_1}\cdot\frac{d}{dy_2}-\frac{d}{dx_2}\cdot\frac{d}{dy_1}\right)^n$ will be unchanged (to a constant coefficient at least) by linear transformations. For the operative symbol just mentioned he uses the abbreviation $\overline{12}^n$. His method then of obtaining the derivatives of a given function U which are unaltered by linear substitution, is to multiply together any number of times quantities $U_1 U_2 U_3$ &c. (U_1 being the function U with x_1, y_1 substituted for x and y); to operate on the product with the product of any number of the symbols $(12)^\alpha (23)^\beta (14)^\gamma$ &c., and *after the differentiation* to drop all the suffixes and suppose all the variables $x_1 y_1$, $x_2 y_2$ equal to x and y. If we operated on a product of quantities *not* the same, suppose $U_1 V_2 W_3$, &c., we should then obtain covariant derivatives not of a single function, but of a system of different functions. In what follows we only speak of derivatives of a single function,

and therefore drop all mention of the quantity operated on, and write $(12)^n$ instead of at full length $(12)^n U_1 U_2$. The principal equation used for transforming these derivatives is the following, given by Mr. Cayley (*Journal*, vol. VIII. p. 121) and which is easily seen to be identically true,

$$\Xi_1(23) + \Xi_2(31) + \Xi_3(12) = 0 \ldots\ldots\ldots\ldots(1),$$

where $$\Xi_1 = x\frac{d}{dx_1} + y\frac{d}{dy_1}, \text{ \&c.}$$

One important result of the above equation is that the expression for any derivative may be so transformed that the highest power of any factor (12) may be an even one. For the derivative is unaltered if we interchange the figures 1 and 2, therefore

$$12^{2m+1}\phi_1 = -12^{2m+1}\phi_2 = \tfrac{1}{2}12^{2m+1}(\phi_1 - \phi_2),$$

and by the help of equation (1) the quantity $\phi_1 - \phi_2$ can be transformed so as to be divisible by (12).

Thus, for example, the derivative $(12)^m(13)$, where m is odd, is transformed as follows,

$$2\Xi_2(12)^m(13) = (12)^m\{\Xi_2(13) - \Xi_1(23)\} = \Xi_3(12)^{m+1};$$

or if n be the degree of U the function operated on,

$$2(n-m)(12)^m(13) = n(12)^{m+1} \ldots\ldots\ldots\ldots(2).$$

It is to be noted here and elsewhere that the possibility of making one of the figures disappear from the symbol for any derivative indicates that that derivative is divisible by U.

Having premised this, I proceed, as the first example, to examine a theorem closely connected with one hereafter to be discussed. "The result of substituting in the reciprocal of the function, $\frac{dU}{dx}$ for ξ and $\frac{dU}{dy}$ for η, is divisible by U," (*Journal*, vol. VII. p. 194). In other words, it is required to calculate $\left(M\frac{d}{dx} - L\frac{d}{dy}\right)^n U$, where $L = \frac{dU}{dx}$ and $M = \frac{dU}{dy}$. Writing for shortness the quantity in question Q_n, it is easy to see that

$Q_2 = (12)(13)$, $Q_3 = (12)(13)(14)$, $Q_4 = (12)(13)(14)(15)$, &c., a particular case of the following more general theorem, "If in any contravariant we substitute $\frac{dU}{dx}$ for ξ &c., the

result will be a covariant whose symbolical expression is obtained by substituting in every factor of the symbol for the contravariant a new figure for the contravariant letter. This will be better understood from the applications of it made afterwards.

Ex. 1. Let us go on now to calculate Q_2 &c. Squaring equation (1) and assembling the terms which only differ by interchange of figures, we have

$$\Xi_3^2(12)^2 - 2\Xi_2\Xi_3(12)(13) = 0,$$

or
$$2(n-1)^2 Q_2 = n(n-1) B_2 U,$$

where I use Mr. Cayley's abbreviation B_n for $(12)^n U$.

Ex. 2. To calculate Q_3. Squaring the equation

$$\Xi_3(12) - \Xi_2(13) = -\Xi_1(23),$$

we have $2\Xi_2\Xi_3(12)(13) = \Xi_3^2(12)^2 + \Xi_2^2(13)^2 - \Xi_1^2(23)^2$,

$$2(n-1)^2 Q_3 = 2\Xi_3^2(12)^2(14) - \Xi_1^2(23)^2(14);$$

and the last term on the right-hand side vanishing identically, we have

$$(n-1)\, Q_3 = n(12)^2(13)U,$$

a result obtained by Mr. Cayley and also by M. Burchardt, (*Journal*, vol. VII. p. 194).

$(12)^2(13)$ is a derivative in its simplest form. When U is of the third degree it is the reciprocal of the first evectant of the discriminant, (*Journal*, vol. I. p. 120).

Ex. 4. To calculate Q_4. Multiply together

$$2\Xi_2\Xi_3(12).(13) = \Xi_3^2(12)^2 + \Xi_2^2(13)^2 - \Xi_1^2(23)^2,$$

$$2\Xi_4\Xi_5(14).(15) = \Xi_5^2(14)^2 + \Xi_4^2(15)^2 - \Xi_1^2(45)^2,$$

and assembling the identical terms

$$4(n-1)^4 Q_4 = \Xi_1^4(23)^2(45)^2 - 4\Xi_1^2\Xi_3^2(12)^2(45)^2 + 4\Xi_2^2\Xi_5^2(12)^2(14)^2,$$

$$= -3n(n-1)(n-2)(n-3) B_2^2 U + 4n^2(n-1)^2 U^2(12)^2(13)^2.$$

But the derivative $(12)^2(13)^2$ may itself be further reduced as follows. Raise to the fourth power the equation

$$\Xi_3(12) - \Xi_2(13) = \Xi_1(23),$$

and we get

$$\Xi_3^4(12)^4 - 8\Xi_3^3\Xi_2(12)^3(13) + 6\Xi_3^2\Xi_2^2(12)^2(13)^2 = 0;$$

but, by equation (2),

$$2\Xi_3^3\Xi_2(12)^3(13) = \Xi_3^4(12)^4,$$

$$\therefore\ 2(n-2)^2(n-3)^2(12)^2(13)^2 = n(n-1)(n-2)(n-3)UB_4 \ldots (3);$$

$$\therefore\ 4(n-1)^4(n-2)(n-3)\,Q_4$$
$$= -3n(n-1)(n-2)^2(n-3)^2UB_2^3 + 2n^2(n-1)^3U^2B_6.$$

Ex. 5. To calculate Q_5.

Multiply by (16) the product of the first two equations of the last Example, and

$$4(n-1)^4Q_5 = -4\Xi_1^2\Xi_5^2(23)^2(14)^2(16) + 4\Xi_3^2\Xi_5^2(12)^2(14)^2(16),$$

$$\text{or } (n-1)^4Q_5 = -n(n-1)(n-3)(n-4)B_2U\{(12)^2(13)\}$$
$$+ n^2(n-1)^3U^2\{(12)^2(13)^2(14)\}.$$

The last term is further reducible by equation (3), which gives

$$2\Xi_2^2\Xi_3^2(12)^2(13)^2(14) = \Xi_3^4(12)^4(14),$$

$$\text{or } 2(n-2)^2(n-3)^2U^2\{(12)^2(13)^2(14)\}$$
$$= n(n-1)(n-2)(n-3)U^2(12)^4(13).$$

Ex. 6. To calculate Q_6.

Multiply together

$$-2\Xi_2\Xi_3(12)(13) = \Xi_1^2(23)^2 - 2\Xi_3^2(12)^2,$$
$$-2\Xi_4\Xi_5(14)(15) = \Xi_1^2(45)^2 - 2\Xi_5^2(14)^2,$$
$$-2\Xi_6\Xi_7(16)(17) = \Xi_1^2(67)^2 - 2\Xi_7^2(16)^2,$$

and we have

$$-8(n-1)^6Q_6 = n(n-1)(n-2)(n-3)(n-4)(n-5)(B_2^3 - 6B_3^2)U$$
$$+ 12n^2(n-1)^2(n-4)(n-5)U^2B_2(12)^2(13)^2$$
$$- 8n^3(n-1)^3U^3(12)^2(13)^2(14)^2.$$

We have already seen how $(12)^2(13)^2$ is reduced, and it remains to shew how $(12)^2(13)^2(14)^2$ may also be reduced. We have at once, by equation (3),

$$2(n-2)^2(n-3)^2(12)^2(13)^2(14)^2 = U\Xi_4(12)^4(13)^2 \ldots (4).$$

When U is of the sixth degree, this $(12)^4(13)^2$ is the reciprocal of the first evectant of the invariant called by Mr. Cayley $D_{1,2,0}$; but it is more convenient to express it generally by help of $C_2 = (12)^2(23)^2(31)^2$. For multiply together

$$2\Xi_2\Xi_3(12)(13) = \Xi_2^2(13)^2 + \Xi_3^2(12)^2 - \Xi_1^2(23)^2,$$
$$2\Xi_3\Xi_1(21)(23) = \Xi_3^2(12)^2 + \Xi_1^2(23)^2 - \Xi_2^2(13)^2,$$
$$2\Xi_1\Xi_2(31)(32) = \Xi_1^2(23)^2 + \Xi_2^2(13)^2 - \Xi_3^2(12)^2,$$

and we have

$$-8\Xi_1^2\Xi_2^2\Xi_3^2(12)^2(23)^2(31)^2 = -3\Xi_3^6(12)^6 + 6\Xi_3^4\Xi_2^2(12)^4(13)^2 - 2\Xi_1^2\Xi_2^2\Xi_3^2(12)^2(23)^2(31)^2,$$

or $2\Xi_2^2\Xi_3^2(12)^2(13)^2 = n(n-1)(n-2)(n-3)B_6U - 2(n-4)^2(n-5)^2C_2.$

Thus Q_6 is expressed in the form

$$U(aB_6U^4 + bC_2U^3 + cB_4B_2U^2 + dB_2^3),$$

a, b, c, d being constant coefficients.

In general it may be observed that Q_m will be of the form

$$U(aU + bB_2).$$

A series proceeding in powers of U can easily be made for Q_m by multiplying together

$$\Xi_2 13 = \Xi_3 12 + \Xi_1 23,$$
$$\Xi_2 14 = \Xi_4 12 + \Xi_1 24,$$
$$\Xi_2 15 = \Xi_5 12 + \Xi_1 25,$$
$$\&c.;$$

and observing that the last term in the right hand side is equal with contrary sign to that on the left, we have a series of the form

$$Q_m = A(12)^m + B(12)^{m-1}(13) + C(12)^{m-2}(13)(14) + \&c. + D(12)^2(13)(14)\ldots$$

But then this admits of further reduction.

Ex. 7. To calculate the Hessian of the Hessian of U.

The symbolical expression for this is (*Journal*, I. p. 118)

$$(\alpha\beta)^2(12)^2(34)^2, \text{ where } \alpha = 1 + 2,\ \beta = (3 + 4),$$

or $\{(13) + (14) + (23) + (24)\}^2(12)^2(34)^2 = 4\{(12)^2(13)^2(34)^2\} + 8\{(12)^2(13)(14)(34)^2\} + 4\{(12)^2(34)(13)(24)\}.$

The second term is at once reduced to the first by substituting for (13)(14), when we have

$$2\Xi_3\Xi_4\{(12)^2(13)(14)(34)^2\} = \Xi_4^2(12)^2(13)^2(34)^2 + \Xi_3^2(12)^2(14)^2(34)^2 - \Xi_1^2(12)^2(34)^4,$$

or $2(n-3)(12)^2(13)(14)(34)^2 = 2(n-2)(12)^2(13)^2(34)^2 - (n-2)B_2B_4.$

Again, the third term is also reduced to the first: for multiplying by $(12)^2$ the product of

$$-2\Xi_4\Xi_1(13)(34) = \Xi_1^2(34)^2 + \Xi_4^2(13)^2 - \Xi_3^2(14)^2,$$
$$2\Xi_3\Xi_2(24)(34) = \Xi_2^2(34)^2 + \Xi_3^2(24)^2 - \Xi_4^2(23)^2,$$

and $-4\Xi_1\Xi_2\Xi_3\Xi_4(12)^2(34)^2(13)(24) = \Xi_1^2\Xi_2^2(12)^2(34)^4$
$+ 2\Xi_3^2\Xi_4^2(12)^2(13)^2(24)^2 - 2\Xi_4^4(12)^2(23)^2(31)^2,$

or $-4(n-3)^3(12)^2(34)^2(13)(24) = (n-2)^2(n-3)B_2B_4$
$-2n(n-1)(n-2)UC_2 + 2(n-2)^2(n-3)(12)^2(13)^2(24)^2.$

Lastly, to calculate the first term, we have

$(12)^2\{(12)^2(34)^2\} = (12)^2\{(13)(24)-(14)(23)\}^2$
$= 2(12)^2(13)^2(24)^2 - 2(12)^2(13)(14)(23)(24),$
$\therefore\ 2(12)^2(13)(14)(23)(24) = 2(12)^2(13)^2(24)^2 - B_2B_4.$

But, multiplying by $(12)^2$ the product of the two equations

$2\Xi_3\Xi_4(13)(14) = \Xi_4^2(13)^2 + \Xi_3^2(14)^2 - \Xi_1^2(34)^2,$
$2\Xi_3\Xi_4(23)(24) = \Xi_4^2(23)^2 + \Xi_3^2(24)^2 - \Xi_2^2(34)^2,$

we have

$4\Xi_3^2\Xi_4^2(12)^2(13)(14)(23)(24) = 2\Xi_4^4(12)^2(23)^2(31)^2$
$-2\Xi_3^2\Xi_4^2(12)^2(24)^2(13)^2 + \Xi_1^2\Xi_2^2(12)^2(34)^4,$

or $4(n-2)(n-3)(12)^2(13)(14)(23)(24) = 2n(n-1)C_2U$
$-2(n-2)(n-3)(12)^2(13)^2(24)^2 + (n-2)(n-3)B_2B_4,$

and comparing this with the value previously obtained,

$6(n-2)(n-3)(12)^2(13)^2(24)^2$
$= 3(n-2)(n-3)B_2B_4 + 2n(n-1)UC_2;$

$-12(n-3)^3(12)^2(34)^2(13)(24)$
$= 6(n-2)^2(n-3)B_2B_4 - 4n(n-1)(n-2)UC_2;$

$6(n-3)^2(12)^2(13)(14)(34)^2$
$= 2n(n-1)UC_2.$

Hence the Hessian of the Hessian can be expressed in the form $aB_2B_4 + bUC_2$, a theorem given by Mr. Cayley (*Crelle*, vol. XXXIV. p. 44), and which may also be proved more easily by shewing that it is true when U is of the fourth degree, and therefore, since no differentials higher than the fourth are introduced, must be true in general.

Ex. 8. It is an important question whether the result of elimination between two equations admits of a general symbolical expression. The result of elimination between a simple equation and one of the n^{th} degree is easily seen to be $(1a)(1b)(1c)\ldots(1n)$, where the figures are symbols referring to the equation of the n^{th} degree and the letters to the simple equation. Let us now examine whether a general expression

can be found for the result of elimination between a quadratic and an equation of the n^{th} degree. Resolve the quadratic into its factors and let a and a' refer to the differentials of the separate factors, and the result of elimination is

$$(1a)(1b)(1c)\ldots\ldots(1n)(2a')(2b')(2c')\ldots\ldots(2n');$$

but if A, B, &c. are symbols relating to the quadratic, we have $A = a + a'$, $B = b + b'$, &c.; $(1A)(2A) = (1a2a') + (1a')(2a)$ (since $(1a)^2$ vanishes); and $(1A)^2 = 2(1a)(1a')$.

We shall write this $(1A)(2A) = [1, 0] + [0, 1]$, where the first figure denotes the number of unaccented letters combined with 1, and the second the number combined with 2. In this notation then

$$(1A)(2A)(1B)(2B)\ldots(1N)(2N)$$
$$= [n, 0] + n[(n-1), 1] + \frac{n(n-1)}{1.2}[(n-2), 2] + \&c.$$

Similarly $(1A)^2(2N)^2(1B)(2B)\ldots(1M)(2M)$
$$= 4\big(\quad [(n-1), 1] + (n-2)[(n-2), 2] + \&c.$$
$$(1A)^3(1B)^3(1C)(2C)\ldots(1L)(2L)$$
$$= 16\big(\quad [n-2, 2] + (n-4)\,\&c.$$

Hence the result of elimination, viz. $[n, 0]$, is

$$(1A)(2A)(1B)(2B)\ \&c. - n\frac{(1A)^2(2A)^2(1B)(2B)\ \&c.}{4}$$
$$+ \frac{n(n-3)}{1.2}\frac{(1A)^3(1B)^3(2N)^3(2M)^3(1C)(2C)\ \&c.}{16} - \&c.,$$

the next coefficients being

$$\frac{n(n-4)(n-5)}{1.2.3}, \quad \frac{n(n-5)(n-6)(n-7)}{1.2.3.4}\ \&c.,$$

(These are the coefficients that occur in the expansion of the sum of the inverse n^{th} powers of the roots of a quadratic.)

A similar process may be applied to higher equations, but as it appears likely that the formula, even in the case of a cubic, would be very complicated, I have not thought it worth while to pursue the subject.

Ex. 9. The result of elimination between two quadratics is, by the preceding series,

$$2\{(13)(14)(23)(24)\} - (13)^2(24)^2,$$

where 1, 2 refer to the first, and 3, 4 to the second. But since we have identically

$$(12)(34) = (13)(24) - (14)(23),$$

$$2(13)(14)(23)(24) = 2(13)^2(24)^2 - (12)^2(34)^2,$$

and the result of elimination is

$$(13)^2(24)^2 - (12)^2(34)^2,$$

or, in terms of the coefficients,

$$(ac' + ca' - 2bb')^2 - 4(ac - b^2)(a'c' - b'^2),$$

a form found by Mr. Boole.

Similarly, the result of elimination between a cubic and quadratic is

$$4\{(1a)(1b)(1c)(2a)(2b)(2c)\} - 3(1a)^2(2b)^2(1c)(2c),$$

and $\quad 2(1a)(2b)(1b)(2a) = 2(1a)^2(2b)^2 - (12)^2(ab)^2;$

therefore the result is

$$(1a)^2(2b)^2(1c)(2c) - 2\{(ab)^2(12)^2(1c)(2c)\},$$

and so generally the series can be transformed in powers of $(ab)^2$, that is of $B^2 - AC$.

I proceed now to the case of homogeneous functions of three variables. We use the abbreviations (123) to denote the determinant

$$\frac{d}{dx_1}\left(\frac{d}{dy_2}\frac{d}{dz_3} - \frac{d}{dy_3}\frac{d}{dz_2}\right) + \frac{d}{dx_2}\left(\frac{d}{dy_3}\frac{d}{dz_1} - \frac{d}{dy_1}\frac{d}{dz_3}\right) + \frac{d}{dx_3}\left(\frac{d}{dy_1}\frac{d}{dz_2} - \frac{d}{dy_2}\frac{d}{dz_1}\right),$$

$(\alpha 12)$ to denote the determinant

$$\alpha\left(\frac{d}{dy_1}\frac{d}{dz_2} - \frac{d}{dy_2}\frac{d}{dz_1}\right) + \beta\left(\frac{d}{dz_1}\frac{d}{dx_2} - \frac{d}{dz_2}\frac{d}{dx_1}\right) + \gamma\left(\frac{d}{dx_1}\frac{d}{dy_2} - \frac{d}{dx_2}\frac{d}{dy_1}\right),$$

Ξ_1 denotes $x\dfrac{d}{dx_1} + y\dfrac{d}{dy_1} + z\dfrac{d}{dz_1}$; and $P = \alpha x + \beta y + \gamma z$.

I notice in the first place that $(123)^2$ is six times the Hessian of the function U, and that $(\alpha 12)^2$ is the function called by Mr. Sylvester the bordered Hessian, and is the same as the equation which I have called the polar conic of a line with regard to a curve of the third degree (*Higher Plane Curves*, p. 151). The curve S (*ib.* p. 101) is simply $(\alpha 12)^4$ and is the

evectant of the invariant $(123)^4$, while the curve T (*ib.* p. 102) is $(\alpha 12)^2(\alpha 23)^2(\alpha 31)^2$. Any invariant of a function of two variables becomes a contravariant of a function of three by prefixing a contravariant symbol α in each factor of the symbolical expression for the invariant, and Mr. Cayley has noticed that the discriminant of a function of two variables treated in this manner gives the equation of the corresponding reciprocal curve.

These symbols are transformed by the identical equations

$$\Xi_4(123) = \Xi_1(234) - \Xi_2(134) + \Xi_3(124) \ldots\ldots (A),$$

$$P(123) = \Xi_1(\alpha 23) + \Xi_2(\alpha 31) + \Xi_3(\alpha 12) \ldots . (B),$$

$$(123)(145) + (124)(153) + (125)(134) = 0 \ldots (C),$$

$$(\alpha 12)(\alpha 34) + (\alpha 23)(\alpha 14) + (\alpha 31)(\alpha 24) = 0 \ldots (D).$$

Since the terms in the last are obtained by cyclically permuting 1, 2, 3, it follows that if the whole equation be multiplied by any quantity unchanged by such a permutation, every term must be separately zero. In this manner are proved the equations of which use is made afterwards,

$$(123)(\alpha 23)(\alpha 14) = 0; \quad (123)(\alpha 23)(\alpha 14)(\alpha 45) = 0, \text{ \&c.}$$

I proceed now to investigate by this method the problem of finding the curve which passes through the points of contact of the double tangents to a given curve. It is proved (see Mr. Cayley's paper, *Crelle*, vol. XXXIV., and *Higher Plane Curves*, p. 81) that if we denote by R_n the result of substituting

$$x_1 = \gamma \frac{du}{dy} - \beta \frac{du}{dz}, \quad y_1 = \alpha \frac{du}{dz} - \gamma \frac{du}{dx}, \quad z_1 = \beta \frac{du}{dx} - \alpha \frac{du}{dy},$$

in $\left(x_1 \dfrac{d}{dx} + y_1 \dfrac{d}{dy} + z_1 \dfrac{d}{dz}\right)^n$, then R_n will always be of the form

$$R_n = P_n U + Q_n (\alpha x + \beta y + \gamma z)^n,$$

and that if we form the discriminant of the equation

$$\lambda^{n-2} Q_2 + \lambda^{n-3} \mu \frac{Q_3}{3} + \lambda^{n-4} \mu^2 \frac{Q_4}{3.4} + \text{\&c.},$$

the result will be divisible by $(\alpha x + \beta y + \gamma z)^{(n-2)(n-3)}$, and will give the equation of the double tangent curve required. The object of what follows is to perform the actual calculation of Q_2, Q_3, &c. And first it is easy to see that R_n may be symbolically expressed as follows:

$$R_2 = (\alpha 12)(\alpha 13), \quad R_3 = (\alpha 12)(\alpha 13)(\alpha 14),$$

$$R_4 = (\alpha 12)(\alpha 13)(\alpha 14)(\alpha 15), \text{ \&c.}$$

Ex. 10. To calculate R_2. Square the equation B, and we have

$$P^2(123)^2 = 3\Xi_3^2(\alpha 12)^2 - 6\Xi_2\Xi_3(\alpha 12)(\alpha 13).$$

We shall denote $(\alpha 12)^n$ by G_n and $(123)^n$ by H_n,

then $$6(n-1)^2 R_2 = 3n(n-1) G_2 U - P^2 H_2,$$

$$6(n-1)^2 Q_2 = -H_2.$$

Ex. 11. To calculate R_3. From equation B,

$$-2\Xi_2\Xi_3(\alpha 12)(\alpha 13) = P^2(123)^2 - 2P\Xi_1(123)(\alpha 23) + \Xi_1^2(\alpha 23)^2 - \Xi_2(\alpha 31)^2 - \Xi_3^2(\alpha 12)^2.$$

Multiply by $(\alpha 14)$, two of the terms vanish identically and

$$-2(n-1)^2 R_3 = P^2(123)^2(\alpha 14) - 2n(n-1) U(\alpha 12)^2(\alpha 13)$$

$$-2(n-1)^2 Q_3 = (123)^2(\alpha 14).$$

Ex. 12. To calculate Q_4. In what follows we omit the terms multiplied by U indicated by one or more figures disappearing from the symbolical expression. Multiply together

$$-2\Xi_2\Xi_3(\alpha 12)(\alpha 13) = P^2(123)^2 - 2P\Xi_1(123)(\alpha 23) + \Xi_1^2(\alpha 23)^2 - \Xi_2^2(\alpha 31)^2 - \Xi_3^2(\alpha 12)^2$$

$$-2\Xi_4\Xi_5(\alpha 14)(\alpha 15) = P^2(145)^2 - 2P\Xi_1(145)(\alpha 45) + \Xi_1^2(\alpha 45)^2 - \Xi_4^2(\alpha 51)^2 + \Xi_5^2(\alpha 14)^2,$$

and $$4(n-1)^4 R_4 = P^4(123)^2(145)^2 - 4P^3\Xi_1(123)^2(145)(\alpha 45)$$

$$+ 2P^2\Xi_1^2(145)^2(\alpha 23)^2$$

$$+ 4P^2\Xi_1^2(123)(\alpha 23)(145)(\alpha 45) - 4P\Xi_1^3(123)(\alpha 23)(\alpha 45)^2.$$

But the last two terms destroy each other, since, on substituting in the first of them for $P(145)$,

$$\Xi_1(\alpha 45) + \Xi_4(\alpha 51) + \Xi_5(\alpha 14),$$

the difference vanishes identically. Hence

$$4(n-1)^4 Q_4 = P^2(123)^2(145)^2 - 4(n-3) P(123)^2(145)(\alpha 45) + 2(n-2)(n-3) H_2 G_2.$$

Ex. 13. To calculate the double tangent curve when U is of the fourth degree.

Its equation is (see *Higher Plane Curves*)

$$Q_3^2 = 3Q_2Q_4.$$

Now $$4(n-1)^4 Q_3^2 = (123)^2(\alpha 17)(456)^2(\alpha 48),$$

but $$\Xi_6(\alpha 17) - \Xi_7(\alpha 18) = P(178) - \Xi_1(\alpha 78),$$
$$\Xi_6(\alpha 47) - \Xi_7(\alpha 48) = P(478) - \Xi_4(\alpha 78).$$

Multiplying, we have

$$-2\Xi_7\Xi_6(\alpha 17)(\alpha 48) = P^2(178)(478) - 2P\Xi_1(178)(\alpha 78) + \Xi_1\Xi_4(\alpha 78)^2,$$

or $$-8(n-1)^6 Q_2^2 = P^2(123)^2(456)^2(178)(478)$$
$$-2(n-2)PH_2(123)^2(178)(\alpha 78) + (n-2)^2 H_2^2 G_2.$$

Hence, $2(n-3)Q_2^2 - 3(n-2)Q_2Q_4$ is proportional to

$$2(n-3)(123)^2(456)^2(178)(478) = (n-2)(123)^2(456)^2(178)^2 \ldots (E),$$

which when $n = 4$ reduces by equation A to the single term

$$(123)^2(456)^2(178)(148) = 0.*$$

If it be required to find whether this equation be or be not identical with the form which M. Hesse has given for the same curve (Crelle, XLI. p. 285, and *Higher Plane Curves*, p. 89); first I answer that $(123)^2(456)^2(178)(478)$ *is* identical with the left-hand side of his equation. The function given by M. Hesse is obtained by substituting in the bordered Hessian $\frac{dH}{dx}$ for α, &c., and therefore may be written $(\alpha 78)(\beta 78)(123)^\alpha(456)^\beta$, where $\alpha = 1+2+3$, $\beta = 4+5+6$, and this expanded gives only terms of the form $(123)^2(456)^2(178)(478)$. But the right-hand side of my equation is *not* identical with that of M. Hesse. His right-hand side is found by operating on H with $\frac{d}{dx}$ for α, &c., substituted in the bordered Hessian, that is to say $(\alpha 45)^2(123)^2$, where $\alpha = 1+2+3$, or

$$(123)^2(145)^2 + 2\{(145)(245)(123)^2\}.$$

There is then apparent disagreement between our results. But I shall prove as follows that the last terms altogether vanish on the supposition $U = 0$. First let us denote by S $(123)(124)(234)(314)$, (it is in fact Aronhold's invariant); then from equation A

$$US = 3\{(125)(124)(234)(314)\},$$

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* At first I imagined that this equation might be represented as a symmetrical function of the curve and the Hessian

$$(123)(234)\,UUHH,$$

the first two figures referring to the curve and the second two to the Hessian. It would be so if the sign had been + in equation E. It is worth mentioning, however, that the points of contact of common tangents to two conics lie on a conic, whose equation is

$$(123)(234)\,UUVV = 0.$$

and $(125)(124)(234)(134) = (125)^2(134)(234)$
$+ 2(245)(125)(234)(134).$

But multiply together the two identical equations

$$(245)(134) - (124)(345) - (145)(234) = 0,$$
$$(245)(123) - (124)(235) + (125)(234) = 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and $-3(125)^2(134)(234) + 6\{(125)(245)(134)(234)\} = 0.$

Therefore the quantity $(123)^2(145)(245)$ reduces to US and therefore vanishes on the supposition $U = 0$. It is to be noted that these are the only terms which exist when U is of the third degree.

The omission then of these superfluous terms renders the actual calculation of the quantity $(123)^2(145)^2$ much more easy in the form I have given it, than in that given by M. Hesse. In fact we have only to square the bordered Hessian $(\alpha 12)^2$ and in it write $\frac{d}{dx}$ for α, &c.

As a step to the consideration of the corresponding problems for the fifth and sixth degrees, I add the values of Q_5 and Q_6, which however are probably capable of further reduction:

$$4(n-1)^4 Q_5 = P^2(123)^2(145)^2(\alpha 16) - 4(n-4)P(123)^2(145)(\alpha 45)(\alpha 16)$$
$$+ 2(n-3)(n-4)G_2(123)^2(\alpha 14)$$
$$+ 4(n-3)(n-4)(123)(\alpha 23)(145)(\alpha 45)(\alpha 16)$$

$$-8(n-1)^6 Q_6 = P^4(123)^3(145)^3(167)^2$$
$$- 6P^3(n-5)(123)^3(145)^2(167)(\alpha 67)$$
$$+ 3P^3(n-4)(n-5)G_2(123)^2(145)$$
$$+ 12P^2(n-4)(n-5)(123)^2(145)(\alpha 45)(167)(\alpha 67)$$
$$- 8P(n-3)(n-4)(n-5)(123)(\alpha 23)(145)(\alpha 45)(167)(\alpha 67)$$
$$- 12P(n-3)(n-4)(n-5)G_2(123)^2(145)(\alpha 45)$$
$$+ 5(n-2)(n-3)(n-4)(n-5)G_2^2 H_2.$$

I proceed to give an illustration of the application of this method to a different problem. Mr. Sylvester observed that a curve of the fourth degree has got a covariant conic section of the fifth degree in the coefficient of the original equation. I noticed myself that it has got another covariant conic of the eighth degree in the coefficients which I ascer-

tained to be distinct from the former. The rules for the calculation of each leading to very laborious processes, it remains to be seen whether any simplification is gained by the use of this method. The first conic is obtained by operating on H with $\frac{d}{dx}$ and substituted for α, &c. in G_4, that is $\{(145)+(245)+(345)\}^4(123)^2$; and since the curve is only of the fourth degree this reduces to the two terms $(145)^2(245)^2(123)^2$ and $(123)^2(145)^2(245)(345)$. Here then it appears we have *two* covariant conics whose coefficients contain those of the original equation in the fifth degree. It remains to be seen whether they are the same or different, but on this question I cannot pronounce at present. The calculation of the former is comparatively easy. It is effected by substituting $\frac{d}{dx}$ for α, &c. in the contravariant conic

$$(\alpha 12)^2(134)^2(234)^2,$$

which involves the coefficients only in the fourth degree.

Sept. 23, 1853.

POSTSCRIPT.

Ex. 14. To find the discriminant of a function of two variables of the fourth degree.

The following is a general process by which invariants may be formed of the same degree in the coefficients of the equation, as the discriminant. Take two sets of $(n-1)$ symbols, 1, 3, 5, &c., 2, 4, 6, &c., and form a product of $(n-1)$ factors, each factor containing one of each set. Thus, in the present instance, we form $(12)(34)(56) = \mathfrak{A}$. Then, by cyclically permuting one of the sets of the symbols, we form $(n-1)$ products in all. Thus, in the present instance, we have

$$(14)(36)(52) = \mathfrak{B}, \quad (16)(32)(54) = \mathfrak{C}.$$

Then any function of the n^{th} degree of these quantities $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c. will be an invariant of the given equation, and of the same degree in the letters as the discriminant, and will vanish identically if the function have more than $\frac{1}{2}n$ equal factors. The two difficulties in the general theory are, first, to ascertain which of these functions *is* the discriminant, and secondly, to discover the relations which exist between the different invariants which may be so formed, and to find the simplest form to which they may be reduced. Thus, for instance, when n is even it is easy to see that

$\mathfrak{A}^n = 2^k \mathfrak{A}^{n-1} \mathfrak{B}$; $k+1$ being the number of factors in the product $\mathfrak{A}$, &c.

In the case then of an equation of the fourth degree, the different invariants that can be formed are $\mathfrak{A}^4$, $\mathfrak{A}^2\mathfrak{B}^2$, $\mathfrak{A}^2\mathfrak{B}\mathfrak{C}$, omitting $\mathfrak{A}^3\mathfrak{B}$, which we know to be only $\frac{1}{4}\mathfrak{A}^4$. The first problem is to form a combination of these invariants, which will vanish identically if the given function be of the form u^2v, where u is of linear dimensions. To do this, suppose the function to be of this form; substitute for 1, $a+\alpha$ (a referring to v and α to u^2) for 2, $b+\beta$, &c., and examine the effect of this substitution on each of the invariants in question. It is obvious that the terms in which either a or α enter above the second degree vanish; and denoting by M the condition $(a\alpha)^2$, that u should be a factor in v, we can find without much trouble that on this substitution we should have

$$\mathfrak{A}^4 = 216M^6, \quad \mathfrak{A}^2\mathfrak{B}^2 = 66M^6, \quad \mathfrak{A}^2\mathfrak{B}\mathfrak{C} = 42M^6;$$

and hence that the discriminant may be expressed by any of the three equations

$$\frac{\mathfrak{A}^4}{36} = \frac{\mathfrak{A}^2\mathfrak{B}^2}{11} = \frac{\mathfrak{A}^2\mathfrak{B}\mathfrak{C}}{7}.$$

Next, for the reduction of these quantities, $\mathfrak{A}^4$ obviously $= B_4^3$. Multiply together the three identical equations,

$$2\{(12)(34)(14)(32)\} = (12)^2(34)^2 + (14)^2(32)^2 - (13)^2(24)^2,$$
$$2\{(34)(56)(36)(54)\} = (34)^2(56)^2 + (36)^2(54)^2 - (35)^2(46)^2,$$
$$2\{(56)(12)(52)(16)\} = (56)^2(12)^2 + (52)^2(16)^2 - (51)^2(26)^2,$$

and
$$8\mathfrak{A}^2\mathfrak{B}\mathfrak{C} = B_4^3 + 4\mathfrak{A}^2\mathfrak{B}^2 - 4C_2^2.$$

Again,

$$C_2^2 = (15)^2(35)^2(46)^2(26)^2\{(12)(34) + (14)(23)\}^2 = 2\mathfrak{A}^2\mathfrak{B}^2$$
$$+ 2[(15)^2(35)^2\{(14)(26)(12)(46)\}\{(34)(26)(23)(46)\}];$$

but $2\{(14)(26)(12)(46)\} = (12)^2(46)^2 + (14)^2(26)^2 - (16)^2(24)^2,$

$-2\{(34)(26)(23)(46)\} = (23)^2(46)^2 + (34)^2(26)^2 - (36)^2(24)^2;$

hence
$$2C_2^2 = 6\mathfrak{A}^2\mathfrak{B}^2 - \tfrac{3}{2}B_4^3.$$

Hence the expression for the discriminant becomes

$$B_4^3 = 6C_2^2,$$

and we have
$$\mathfrak{A}^2\mathfrak{B}^2 = \tfrac{1}{4}B_4^3 + \tfrac{1}{3}C_2^2,$$
$$\mathfrak{A}^2\mathfrak{B}\mathfrak{C} = \tfrac{1}{4}B_4^3 - \tfrac{1}{3}C_2^2.$$

It is to be observed, that we have here made use of one equation more than was necessary for the solution of the problem.

GENERAL CONSIDERATIONS ON CONIC SECTIONS IN DOUBLE CONTACT WITH EACH OTHER.

By Prof. Steiner.

Section X.

In the former memoir,* the properties of all conic sections touching each of two given fixed circles twice were considered. I propose here to append more general considerations, where, instead of two circles, any two conic sections in fixed position will be given, (which may again be represented by A^2 and B^2), and where, in a similar manner, the properties of all conic sections touching each of the given ones twice will be considered.

I. In order to have a definite case (figure) in view, let us imagine, or construct, two ellipses A^2 and B^2, intersecting each other in four points r, s, t, u, and consequently, having four real tangents R, S, T, U in common; let R be the tangent from whose points of contact two elliptical arcs lead immediately to the intersection r, and similarly for the rest. The four intersections form a complete quadrangle $rstu$, and the four tangents a complete quadrilateral $RSTU$. Let us distinguish the three pairs of opposite sides of the first, and their intersections, as well the three pairs of opposite angles of the latter, and its three diagonals, by the following letters:

Side	$rs = \Xi$	and	$tu = \Xi_1$;	Intersection	$\Xi\Xi_1 = x$.
"	$rt = \mathrm{H}$	and	$su = \mathrm{H}_1$;	"	$\mathrm{HH}_1 = y$.
"	$ru = \mathrm{Z}$	and	$st = \mathrm{Z}_1$;	"	$\mathrm{ZZ}_1 = z$.
Angle	$RS = \xi$	and	$TU = \xi_1$;	Diagonal	$\xi\xi_1 = X$.
"	$RT = \eta$	and	$SU = \eta_1$;	"	$\eta\eta_1 = Y$.
"	$RU = \zeta$	and	$ST = \zeta_1$;	"	$\zeta\zeta_1 = Z$.

The intersections x, y, z of the three pairs of opposite sides of the quadrangle are the common triplet of conjugate poles of the two ellipses, and the three diagonals X, Y, Z of the quadrilateral are the common triplet of conjugate polars of the same: hence it follows that

Intersection $XY = z$, $XZ = y$; $YZ = x$, and

Line $xy = Z$, $xz = Y$; $yz = X$.

Further x, ζ, y and ζ_1; x, η, z and η_1; y, ξ, z and ξ_1 are four

* *Math. Jour.*, vol. VIII. p. 227.

harmonical points, as well as X, Z, Y and Z_1; X, H, Z and H_1; Y, Ξ, Z and Ξ_1 four harmonical lines.

Employing this, together with the notation and nomenclature already introduced in the former memoir, the properties under consideration may be thus expressed.

II. All conic sections C^2 in double contact with each of the given ellipses A^2 and B^2, in consequence of their relation to the three poles x, y, z, may be divided into three distinct throngs $Th(C_x^2)$, $Th(C_y^2)$, and $Th(C_z^2)$ (§ *VIII.*) which in general, however, have like properties, so that, for brevity's sake, we need here only consider the $Th(C_x^2)$.

(1) "*If the curve C_x^2 touches the ellipse A^2 in the points p and p_1, and the ellipse B^2 in the points q and q_1, the chords of contact pp_1 and qq_1 pass through the pole x, and are always conjugate harmonical to the opposite sides Ξ and Ξ_1.*" And conversely: "*If through the pole x any two lines G and H be drawn conjugate harmonical to the sides Ξ and Ξ_1, they will intersect the ellipses A^2 and B^2 respectively in certain points p, p_1 and q, q_1, so that in these points the ellipses will be touched by some curve C_x^2; and alternately, H will intersect A^2, and G will intersect B^2 in certain points p°, p_1° and q°, q_1°, so that in these same points A^2 and B^2 will be touched by some other curve C_x^2.*"

(2) "*If between each two pairs of corresponding points of contact p and p_1, q and q_1, the four alternate chords pq, pq_1, p_1q and p_1q_1 be drawn (§ VIII. 5), they will all be tangents to a certain conic section X^2, inscribed in the quadrilateral $RSTU$, and touching the two opposite sides Ξ and Ξ_1: (inasmuch as the latter as well as the tangents R, S, T, U are themselves special alternate chords).*" And further: "*The two pairs of tangents, drawn from the two intersections of the polar X with the ellipse B^2 (or A^2) to the conic section X^2, touch the latter in the same points in which it is intersected by the other ellipse A^2 (or B^2).*"

(3) "*If through the four points of contact p and p_1, q and q_1, any arbitrary conic section D^2 be drawn, it will intersect the given curves A^2 and B^2 in four new points p° and p_1°, q° and q_1°, so that in these same points A^2 and B^2 will be touched by some other curve C_x^2.*" And conversely: "*The eight points of contact of any two curves C_x^2 with the ellipses A^2 and B^2 lie always in some conic section D^2.*"

"*All curves C_x^2 have, in common, x and Ξ as pole and polar. Of the common secants to any two curves C_x^2, one pair, G and H, always passes through the pole x, and they are always conjugate harmonical to Ξ and Ξ_1.*"

(4) "*Every four points of contact* p, p_1, q, q_1 *lie on the one hand with the angles* r *and* s *in a conic section* M^2, *and on the other hand with the angles* t *and* u *in a conic section* M_1^2. *All the conic sections* M^2, *hereby defined, touch each other in the points* r *and* s, *so that they there have common tangents* $\mathfrak{P}$ *and* $\mathfrak{Z}$, *and the common chord of contact* $rs = \Xi$, *thus forming a special pencil of curves* $Pl(M^2)$. *Let* m *be the intersection of the tangents* $\mathfrak{P}$ *and* $\mathfrak{Z}$; *it is situated in the polar* Ξ, *and* m *and* Ξ *are pole and polar in reference to all* M^2 *or* $Pl(M^2)$. *Let* a *and* b *be the poles of the side* Ξ *in reference to* A^2 *and* B^2, *these are also situated in* X, *and let* c *be the intersection of* X *and* Ξ: *then the four points* a, m, b, c *are harmonical, so that the pole* m *is defined by the three points* a, b, c, *which are to be considered as given, and by means of* m, *the tangents* $\mathfrak{P}$ *and* $\mathfrak{Z}$ (= mr *and* ms) *are also determined. In an exactly similar manner, all conic sections* M_1^2 *touch each other in the points* t *and* u, *so that they there have common tangents of contact* $\mathfrak{T}$ *and* Υ, *and the common chord of contact* $tu = \Xi_1$; *hence they form a special pencil of curves* $Pl(M_1^2)$; *and further, the intersection* m_1 *of* $\mathfrak{T}$ *and* Υ, *together with the poles* a_1 *and* b_1 *of the side* Ξ_1 *in reference to the ellipses* A^2 *and* B^2, *lie all in the same polar* X; *and if* c_1 *be the intersection of* X *with* Ξ_1, *then the four points* a_1, m_1, b_1, c_1 *are harmonical; therefore from* a_1, b_1, c_1 *the pole* m_1 *is determined, and from the latter the tangents* $\mathfrak{T}$ *and* Υ." "*The two pairs of tangents* $\mathfrak{P}$ *and* $\mathfrak{Z}$, $\mathfrak{T}$ *and* Υ, *thus determined, also touch the above conic section* X^2, *the locus of all alternate chords* (2); *and indeed* $\mathfrak{P}$ *and* $\mathfrak{Z}$ *touch it in its intersections with the side* Ξ_1, *and similarly* $\mathfrak{T}$ *and* Υ *touch it in its intersections with the side* Ξ, *so that alternately, with respect to* X^2, m *is the pole of* Ξ_1, *and* m_1 *the pole of* Ξ." If these two pairs of tangents are assumed as given, we can conversely assert that "*Every conic section* M^2, *which touches the lines* $\mathfrak{P}$ *and* $\mathfrak{Z}$ *in the points* r *and* s, *intersects the given curves* A^2 *and* B^2 *in four points* p *and* p_1, q *and* q_1, (*as well as in* r *and* s), *which are the points of contact of some curve* C_x^2 *with them.*" "*The four mutual intersections of every two* C_x^2 *are always situated in a conic section* M^2 (*which touches* $\mathfrak{P}$ *and* $\mathfrak{Z}$ *in* r *and* s); *and conversely, every conic section* M^2 *intersects any curve* C_x^2 *in four points, through which some other curve* C_x^2 *also passes. The same applies to the conic sections* M_1^2."

III. (1) "*Let* P *and* P_1, Q *and* Q_1 *be the tangents of contact of a curve* C_x^2 *with the given curves* A^2 *and* B^2, *then the intersections* $PP_1 = p$ *and* $QQ_1 = q$ *lie always in the polar* X,

and are always conjugate harmonical to the angles ξ *and* ξ_1." And conversely: "*If in the polar X, any two points g and h are taken, conjugate harmonical to the opposite angles* ξ *and* ξ_1, *then the tangents P and* P_1, *Q and* Q_1 *drawn therefrom to the curves* A^2 *and* B^2 *are at the same time tangents of contact of these curves with some curve* C_x^2; *and similarly, the tangents* P° *and* P_1°, Q° *and* Q_1° *drawn (alternately) from h to* A^2, *and from g to* B^2 *are at the same time the tangents of contact of some other curve* C_x^2 *with* A^2 *and* B^2."

(2) "*Every two pairs of corresponding tangents of contact P and* P_1, *Q and* Q_1 *have four alternate intersections PQ and* PQ_1, P_1Q *and* P_1Q_1; *the locus of all these alternate intersections is a certain conic section* Ξ^2, *which circumscribes the quadrangle rstu, and also passes through the opposite angles* ξ *and* ξ_1." And further: "*The tangents drawn from the pole x to the ellipse* B^2 (*or* A^2) *intersect the conic section* Ξ^2 *in the same points in which it is touched by the four tangents which it has in common with the other ellipse* A^2 (*or* B^2)."

(3) "*If any arbitrary conic section* D^2 *be drawn, touching the four tangents of contact P and* P_1, *Q and* Q_1, *it will have, in common with the given curves* A^2 *and* B^2, *two more pairs of tangents* P° *and* P_1°, Q° *and* Q_1°, *which will always be the tangents of contact of some other curve* C_x^2 *with* A^2 *and* B^2." And conversely: "*The eight tangents of contact of any two curves* C_x^2 *with the curves* A^2 *and* B^2 *are always tangential to some conic section* D^2." "*Of the mutual intersections of the tangents common to any two curves* C_x^2, *one pair, g and h, are always situated in the polar X, and are always conjugate harmonical to the opposite angles* ξ *and* ξ_1."

(4) "*Every four tangents of contact P,* P_1, *Q,* Q_1 *are on the one hand, together with the tangents R and S, touched by a certain conic section* M^2, *and on the other hand, together with the tangents T and U, are touched by a conic section* M_1^2. *All the conic sections* M^2, *hereby defined, touch the tangents R and S in the same points* ρ *and* σ, *and hence they there touch each other also, so as to have* $\rho\sigma = \mathrm{M}$ *as there common chord of contact, and* ξ *and* M *as their common pole and polar; they form consequently a special pencil of curves* $Pl(\mathrm{M}^2)$. *The chord of contact* M *passes through the pole x, as also do the polars of* ξ *in reference to* A^2 *and* B^2, *which we may call* A *and* B; *and if further, the line* $x\xi$ *be* Γ, *then the four lines* A, M, B, Γ *are harmonical, consequently* M *is defined by the three others, which may be considered as given, and* M *determines the points*

ρ and σ, inasmuch as they are its intersections with R and S. In an exactly similar manner, the conic sections M_1^2, *or* $Pl(M_1^2)$, *touch the tangents T and U in two certain points τ and υ, &c., &c.*" "*The two pairs of points of contact ρ and σ, τ and υ thus defined are situated in the above conic section* Ξ^2, *the locus of alternate intersections* (2), *and in fact, the tangents in ρ and σ to* Ξ^2 *pass through the angle* ξ_1, *and the tangents to the same conic section in τ and υ both pass through the angle* ξ, *so that in reference to* Ξ, *inversely,* M *is the polar of* ξ_1, *and* M_1 *the polar of* ξ." Assuming the four points ρ and σ, τ and υ as known, we may conversely assert that, "*Every conic section* M^2 (*or* M_1^2) *which touches the tangents R and S* (*or T and U*) *in the points ρ and σ* (*or τ and υ*) *has, in common with the given curves* A^2 *and* B^2, *two other pairs of tangents P and* P_1, *Q and* Q_1 *which at the same time are tangents of contact of some curve* C_x^2 *with* A^2 *and* B^2." And further: "*The common tangents of any two curves* C_x^2 *always touch one of the conic sections* M^2, *as well as one of the* M_1^2; *and conversely: Every conic section* M^2, *or* M_1^2, *has four tangents in common with every curve* U_x^2, *which are also touched by some other curve* C_x^2."

IV. "*If in the two points where the side* H (I.) *is intersected by any curve* C_x^2, *the tangents to the latter be drawn, one of them will always pass through the angle ζ, the other through* ζ_1; *and similarly, of the two tangents to the same curve in its two points of intersection with side* H_1, *the one always passes through ζ, the other through* ζ_1." Or conversely: "*If from the angle ζ, or* ζ_1, *the two tangents to a curve* C_x^2 *be drawn, the point of contact of the one tangent will always be in the side* H, *and that of the other in the side* H_1." "*And in like manner, of the two tangents to any curve* C_x^2 *in its points of intersection with the side* Z, *or* Z_1, *the one always passes through the angle η, the other through* η_1; *or conversely, Of the points of contact made by two tangents drawn from the angle η, or* η_1, *to any curve curve* C_x^2, *one is situated in the side* Z, *the other in the side* Z_1."

As before remarked, all the foregoing theorems, referring to the $Th(C_x^2)$, are, in analogous manner, applicable to the throngs $Th(C_y^2)$ and $Th(C_z^2)$, so that each theorem is virtually thrice present. The elements which are each time involved may be easily recognised. For instance, in theorem (IV.) where unlike-named elements are involved, the combination is,

$$Th(C_x^2) \text{ with } \begin{cases} \text{H, } H_1 \text{ and } \zeta,\ \zeta_1; \\ \text{Z, } Z_1 \text{ and } \eta,\ \eta_1; \end{cases}$$

and hence, the combinations for the two other cases are

$$Th(C_y^2) \text{ with } \begin{Bmatrix} \Xi, \ \Xi_1 \text{ and } \zeta, \ \zeta_1 ; \\ \mathrm{Z}, \ \mathrm{Z}_1 \text{ and } \xi, \ \xi_1 ; \end{Bmatrix};$$

and

$$Th(C_x^2) \text{ with } \begin{Bmatrix} \Xi, \ \Xi_1 \text{ and } \eta, \ \eta_1 ; \\ \mathrm{H}, \ \mathrm{H}_1 \text{ and } \xi, \ \xi_1 ; \end{Bmatrix}.$$

There are theorems, however, which at the same time refer to two throngs; for example, the following:

V. (1) "*All poles of the side* Ξ, *in reference to the* $Th(C_y^2)$ *as well as in reference to the* $Th(C_x^2)$, *together with its two poles a and b, in reference to* A^2 *and* B^2, *lie entirely in one and the same conic section* M_x^2, *which is one of the above pencil* $Pl(M^2)$ (II. 4), *therefore touches the lines* P *and* Σ *in the angular points r and s; it passes also through the two pairs of opposite angles* η *and* η_1, ζ *and* ζ_1 *of the quadrilateral RSTU.*" "*In like manner, all poles of the side* Ξ_1 *in reference to the* $Th(C_y^2)$ *and* $Th(C_x^2)$, *together with its poles* a_1 *and* b_1 *in reference to* A^2 *and* B^2, *are situated in a certain conic section* M_{1x}^2, *which belongs to the above pencil* $Pl(M_1^2)$ (II. 4), *therefore touches the lines* T *and* Υ *in the angular points t and u, and passes also through the same opposite angles* η *and* η_1, ζ *and* ζ_1." This theorem also is thrice present, *i.e.* with respect to each of the three pairs of opposite sides of the quadrangle *rstu*, and to the two throngs of unlike name with the side each time involved.

(2) "*All polars of the angle* ξ *in reference to the* $Th(C_y^2)$ *and* $Th(C_x^2)$, *together with its polars* A *and* B *in reference to* A^2 *and* B^2, *are tangents to a certain conic section* M_x^2, *which latter belongs to the above pencil* $Pl(\mathrm{M}^2)$ (III. 4), *therefore touches the tangents R and S in the points* ρ *and* σ, *and further, is also tangential to the two pairs of opposite sides* H *and* H_1, Z *and* Z_1 *of the quadrangle rstu. And in similar manner, all polars of the angle* ξ_1 *in reference to the* $Th(C_y^2)$ *and* $Th(C_x^2)$, *together with its polars* A_1 *and* B_1 *in reference to* A^2 *and* B^2, *touch a certain conic section* M_{1x}^2, *which latter belongs to the above pencil* $Pl(\mathrm{M}_1^2)$, *and therefore touches the tangents T and U in the points* τ *and* υ, *as well as the sides* H *and* H_1, Z *and* Z_1."

VI. *Note.* The above properties are true for the case supposed, that is to say, when the four mutual intersections, as well as the four common tangents of the given conic sections A^2 and B^2 are real, whereby, however, the latter need not be ellipses, but may be any kind of conic sections.

From the case supposed we can pass to other cases, where any portion of the above elements is imaginary. The most essential of these cases are the following three. If the relative situation of the given arbitrary conic sections A^2 and B^2 be such, that either (1) the four intersections r, s, t, and u only are real, the four common tangents being, on the contrary, imaginary; or (2) the four common tangents are real, the four intersections being imaginary; or (3) only two intersections, and two common tangents are real. In each of these three cases, a portion of the elements described above (I.) will always be imaginary; hence in the above properties and theorems, corresponding, more or less considerable changes must be introduced in a similar manner as was before done in § *VIII.* For example, if the opposite sides Ξ and Ξ_1 become imaginary, but their intersection, the pole x, remains real, the theorem (II. 1) will be so changed, that all the pairs of chords of contact pp_1 and qq_1 will form an *elliptic* system of rays, instead of a *hyperbolic* one, as before, &c.

Section XI.

I. The following special theorems, included in the above general theorems (§ X.), may be here particularized.

(1) "*If in a complete quadrilateral RSTU, any two conic sections* A^2 *and* B^2 *be inscribed, the eight points of contact are always situated in some third conic section* D^2." And, "*If through the four points* ρ, σ, τ, *and* υ, *in which any conic section* A^2 *touches the sides R, S, T, and U of the quadrilateral, any conic section* D^2 *be drawn, the same will intersect the sides in four other points* ρ_1, σ_1, τ_1, *and* υ_1, *so that in these latter points the said sides are also touched by some other conic section* B^2." * Further: "*The four mutual intersections r, s, t, u of any two conic sections* A^2 *and* B^2, *inscribed in the same quadrilateral RSTU, together with any one of the three pairs of opposite angles* ξ *and* ξ_1, η *and* η_1, *or* ζ *and* ζ_1 *of the same, are situated in a conic section* Ξ^2, H^2, *or* Z^2." And further: "*Of the* 8 *points of contact* $(\rho, \sigma, \tau, \upsilon;\ \rho_1, \sigma_1, \tau_1, \upsilon_1)$ *of any two conic sections* A^2 *and* B^2, *inscribed in the quadrilateral,* 12 *times* 4 *lie, together with two of the four intersections r, s, t, u of these conic sections, in a certain conic section* M^2 *(or* M_1^2*); and these* 12 *conic sections arrange themselves*

* The analogous theorems to the above two, but in reference to the trilateral I have already proved in *Gergonne's Annales des Math.*, tom. XIX. or XX. 1828.

in 6 *pairs which touch each other twice, for through each two of the four intersections r, s, t, u pass two conic sections* M^2, *which there touch each other. The several* 6 *points situated together in some conic section* M^2, *are*

$$\begin{vmatrix} \rho, & \sigma, & \rho_1, & \sigma_1 \\ \tau, & \upsilon, & \tau_1, & \upsilon_1 \end{vmatrix} \text{ with } \begin{vmatrix} r, & s \\ t, & u \end{vmatrix}; \quad \begin{vmatrix} \rho, & \tau, & \rho_1, & \tau_1 \\ \sigma, & \upsilon, & \sigma_1, & \upsilon_1 \end{vmatrix} \text{ with } \begin{vmatrix} r, & t \\ s, & u \end{vmatrix};$$

$$\begin{vmatrix} \rho, & \upsilon, & \rho_1, & \upsilon_1 \\ \sigma, & \tau, & \sigma_1, & \tau_1 \end{vmatrix} \text{ with } \begin{vmatrix} r, & u \\ s, & t \end{vmatrix},$$

that is, both the four points in the first parenthesis, together with each of the two pairs in the second parenthesis, are situated in some conic section M^2.

(2) "*If a quadrangle rstu is circumscribed by any two conic sections* A^2 *and* B^2, *their* 8 *tangents in the four angles r, s, t, and u always touch some third conic section* D^2." And, "*If the quadrangle is circumscribed by any conic section* A^2, *and its four tangents* P, Σ, T, *and* Υ, *in the angles r, s, t, and u, are touched by any other conic section* D^2, *then there are four tangents* P_1, Σ_1, T_1, *and* Υ_1 *to the latter, passing through the same angles of the quadrangle, which are always touched in those angles by some third conic section* B^2." Further: "*The four common tangents R, S, T, U of any two conic sections* A^2 *and* B^2 *circumscribing the same quadrangle rstu, together with any one of the three pairs of opposite sides of the quadrangle* Ξ *and* Ξ_1, H *and* H_1, *or* Z *and* Z_1 *are touched by some conic section* X^2, Y^2, *or* Z^2." And further: "*Of the* 8 *tangents* (P, Σ, T, Υ; P_1, Σ_1, T_1, Υ_1) *in the angles of a quadrangle rstu to any two conic sections* A^2 *and* B^2 *circumscribing the same,* 12 *times* 4, *together with two of the four common tangents R, S, T, U, are touched by some conic section* M^2; *and further, these* 12 *conic sections* M^2 *arrange themselves in* 6 *pairs, touching each other twice, which have the four common tangents R, S, T, and U, taken in pairs, as tangents of contact.*"

II. From the above considerations (§ X.), we can also pass to the special cases where the given conic sections A^2 and B^2, separately considered, consist of a pair of points or lines. In this respect the following five cases may be noticed.

(1) Let B^2 consist of two lines Z and Z_1; then these are a pair of opposite sides of the quadrangle *rstu*, and their mutual intersection is the pole *z*. The four common tangents *R*, *S*, *T*, and *U* coincide in pairs, (*RU*) and (*ST*), and are the two tangents to the curve A^2, passing through *z*. Hence,

two pairs ξ and ξ_1, η and η_1 of the six angles of the former quadrilateral $RSTU$ coincide with each other in the point z, and the remaining two ζ and ζ_1 are the points of contact of the tangents (RU) and (ST); they are situated in the polar Z, and are conjugate harmonical to the poles x and y. Hereby the $Th(C_z^2)$ resolves itself into a pencil of rays around the centre z, *i.e.* each ray (right line) passing through z, conceived as double, is to be regarded as a C_z^2; its intersections with A^2 are at the same time its points of contact with the same, whereas its points of contact with $B^2 = (ZZ_1)$ are all combined in z. The throngs $Th(C_x^2)$ and $Th(C_y^2)$ remain actual curves and retain, with a few suitable modifications, all their former properties.

(2) Let B^2 consist of two points ζ and ζ_1; then these are a pair of opposite angles of the quadrilateral $RSTU$, and are situated in the polar Z. The four intersections r, s, t, and u coincide in pairs, (ru) and (st), in the intersections of $\zeta\zeta_1 = Z$ with the curve A^2; so that two pairs of opposite sides of the quadrangle $rstu$, viz. Ξ and Ξ_1, H and H_1, coincide with Z, and the third pair, Z and Z_1, become the tangents to A^2 in points (ru) and (st); these latter intersect each other in z, and are conjugate harmonical to X and Y. Here the throng $Th(C_z^2)$ consists of all pairs of tangents to the curve A^2 which intersect each other on the line Z. On the other hand, the $Th(C_x^2)$ and $Th(C_y^2)$ contain actual conic sections C^2, which pass through the given points ζ and ζ_1, and touch the given curve A^2 twice.

(3) Let A^2 and B^2 consist each of two lines H and H_1, Z and Z_1; these must be considered as two pairs of opposite sides of the quadrangle $rstu$, their own intersections being the poles y and z, and their alternate intersections the points r, s, t, and u. The four common tangents R, S, T, and U all coincide with the line $yz = X$. The $Th(C_y^2)$ and $Th(C_z^2)$ resolve themselves into pencils of rays around the poles y and z in the same manner as above (1); and the $Th(C_x^2)$ alone contains actual curves, whose chords of contact pass through the pole x; their alternate chords, however, pass in pairs through the poles y and z (§ X. 11.).

(4) Let A^2 and B^2 consist each of a pair of points η and η_1, ζ and ζ_1; these must then be considered as opposite angles of the quadrilateral $RSTU$, the lines $\eta\eta_1$ and $\zeta\zeta_1$ joining them, as the polars Y and Z, and the two pairs of lines joining them alternately, as the common tangents R, S, T, and U. The four intersections r, s, t, and u all coincide

in the point $x = YZ$. The $Th(C_y^2)$ resolves itself into all pairs of lines which pass through the points ζ and ζ_1, and intersect on Y; and similarly, the $Th(C_z^2)$ consists of all pairs of lines passing through the points η and η_1, and intersecting each other on Z. The $Th(C_x^2)$ continue actual curves, circumscribing the quadrangle $\eta\eta_1\zeta\zeta_1$.

(5) Lastly, let A^2 consist of two points ζ and ζ_1, and B^2 of two lines Z and Z_1; these must be considered as the elements which these letters represent, so that the line $\zeta\zeta_1 = Z$, and the intersection $ZZ_1 = z$. The four common tangents coincide in pairs with the lines $z\zeta$ and $z\zeta_1$, *i.e.* $(RU) = z\zeta$ and $(ST) = z\zeta_1$, and hence the two pairs of opposite angles ξ and ξ_1, η and η_1 coincide in z. Similarly, the four intersections r, s, t, u coincide in pairs (r and u, s and t) with the intersections of Z with Z and Z_1, so that $(ru) = ZZ$, and $(st) = ZZ_1$, and therefore both pairs of opposite sides Ξ and Ξ_1, H and H_1 coincide with Z; their intersections however, the poles x and y, remain still defined, inasmuch as they are conjugate harmonical to ζ and ζ_1, as well as to the intersections ZZ and ZZ_1. The $Th(C_z^2)$ here resolves itself into a pencil of rays around z. On the contrary, the $Th(C_x^2)$ and $Th(C_y^2)$ contain actual curves, which pass through the points ζ and ζ_1, and touch the lines Z and Z_1.

III. Knowing these special cases (II.) as well as the above general one (§ X.), the following problems can be easily treated, and the number of solutions of which they are capable can be at once determined.

(1) *To find a curve C^2, which touches each of two given conic sections A^2 and B^2 twice, and also, either*

α. touches a given line G; or

β. passes through a given point p.

Each of these problems admits of six solutions, and the curves solving them consist of $2C_x^2$, $2C_y^2$ and $2C_z^2$. The four points p, p_1, p_2, and p_3, in which the pair of curves $2C_z^2$ intersect each other, are situated in one of the conic sections M^2 (§ X. II. 4), which is defined by the point p; the intersections p_1, p_2, and p_3 are also determined, for one, p_3, lies in the line xp; again, the line p_1p_3 also passes through x, and further, both lines pp_3 and p_1p_3 are conjugate harmonical to the sides Ξ and Ξ_1.

(2) *To find a curve C^2, which touches a given curve A^2 twice, and also, either*

α. *touches three given lines* Z, Z_1 *and* G *; or*
β. *touches two given lines* Z, Z_1, *and passes through a given point* p *; or*
γ. *touches a given line* G, *and passes through two given points* ζ *and* ζ_1 *; or lastly*
δ. *passes through three given points* ζ, ζ_1, *and* p.

Each of these four problems admits, in general, of six solutions as before (1).

(3) *To find a curve* C^2, *which either*
α. *touches three given lines* Z, Z_1, *and* G, *and passes through two given points* ζ *and* ζ_1 *; or*
β. *touches two given lines* Z *and* Z_1, *and passes through three given points* ζ, ζ_1, *and* p.

In each case there are four solutions.

(4) *To find a curve* C^2, *which either*
α. *touches four given lines* (II. 3), *and passes through a given point* p *; or*
β. *passes through four given points* (II. 4), *and touches a given line* G.

In each case there are two solutions. And lastly:

(5) *To find a curve* C^2, *which either*
α. *touches five given lines ; or*
β. *passes through five given points.*

In each case there is but one solution, hence C^2 is perfectly defined.

Section XII.

Note. The solution of the problem, "*To find a curve* C^2, *which makes a double contact with each of the three given curves* A^2, B^2 *and* D^2," is in general impossible, as is evident from the above (§ X.); it is only possible when the given curves have a certain more intimate relation to each other; as has already been shewn in the memoir so often referred to,* and which may be easily demonstrated as follows:

For, let us suppose the curve C^2 touches each of the given circles A^2, B^2, and D^2 twice; let A, B, and D be the respective chords of contact, and let further x, y, z; x', y', z'; x'', y'', z'' be the common triplets of conjugate poles, and X, Y, Z; X', Y', Z'; X'', Y'', Z'' the common triplets of conjugate polars of the pairs of curves A^2 and B^2, A^2 and D^2, B^2 and D^2; then the chord of contact A must pass through a pole of the first triplet, *e.g.* x, as well as through a pole

* *Crelle's Journal*, Bd. 37, S. 187.

of the second triplet, *e.g.* x', (because A^2 belongs to the first as well as to the second pair of curves) (§ X. II. 1); and then the chords of contact B and D must pass respectively through the same poles, x and x', as well as both, through one and the same pole of the third triplet, *e.g.* x''. Hence the three chords of contact A, B, D must be the sides of a certain triangle whose angles are any three poles, provided there be one from each triplet, as for example the triangle $xx'x''$; by combination, there are 27 such triangles. Further, as x and X, x' and X', and x'' and X'' are each pole and polar, in reference to the curve C^2 (§ X. II. 3), the triangle $xx'x''$ must be perspectively situated with respect to the trilateral $\Xi\Xi'\Xi''$; *i.e.* the three lines which join their angles, taken by pairs in a certain order, must meet in some point p; and the three intersections of the corresponding pairs of sides must be situated in some line P; in other words, if a, b, d be the angles of the trilateral, respectively opposite to the sides X, X', X'', (they are at the same time the poles of the sides A, B, D of the triangle $xx'x''$ in reference to the curve C^2, and respectively so in reference to the given curves A^2, B^2, D^2,) then the three lines ax, bx', dx'' must meet in a certain point p; and the three intersections AX, BX', DX'' must be in a line P (p and P are pole and polar in reference to C^2). These properties are accompanied further by the following. If the four points, in which the three pairs of curves each mutually intersect each other, be respectively r, s, t, u; r', s', t', u'; r'', s'', t'', u'', and their four common tangents respectively R, S, T, U; R', S', T', U'; R'', S'', T'', U'', and the remaining elements be represented in a like manner; then, the pairs of sides Ξ and Ξ_1, Ξ' and Ξ_1', Ξ'' and Ξ_1'' pass respectively through the poles x, x', x'', and the pairs of angles ξ and ξ_1, ξ' and ξ_1', ξ'' and ξ_1'' are situated respectively in the polars X, X', X''; of the first, four times three must intersect in a point (*e.g.* $\Xi\Xi'\Xi''$, $\Xi\Xi_1'\Xi_1''$, $\Xi'\Xi_1\Xi_1''$, $\Xi''\Xi_1\Xi_1'$), and of the last, four times three must be in a line (*e.g.* $\xi\xi'\xi''$, $\xi\xi_1'\xi_1''$, $\xi'\xi_1\xi_1''$, $\xi''\xi_1\xi_1'$).

If, therefore, a curve C^2 touch each of the three given curves A^2, B^2, D^2 twice, the elements of the latter must, among others, have the above properties. As these properties, however, condition each other, or are themselves dependent upon each other, the condition under which C^2 is possible, limits itself solely to any one portion of the same, viz.

"*The curve* C^2, *which shall touch each of the three given curves* A^2, B^2, D^2 *twice, is possible if*

(1) *of the 27 triangles, whose corners are any three poles, selected one out of each triplet, any one (as $xx'x''$) is perspectively situated with respect to its corresponding trilateral $(XX'X'')$; or if*

(2) *of the sides of the three quadrangles $rstu$, $r's't'u'$, $r''s''t''u''$ any three, one being selected from each quadrangle (as $\Xi\Xi'\Xi''$), meet in one point p; or lastly, if*

(3) *of the angles of the three quadrilaterals $RSTU$, $R'S'T'U'$, $R''S''T''U''$ any three, one being selected from each quadrilateral (as $\xi\xi'\xi''$), are situated in a right line P."*

ON CLAIRAUT'S THEOREM, AND SOME MATTERS CONNECTED WITH IT.

By MATTHEW COLLINS, B.A.,
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To prove that a body has three *principal* axes relative to any point, *i.e.* three right lines meeting at this point, at right angles to each other, and such that

$$\int xy\,dm = 0, \quad \int xz\,dm = 0, \quad \text{and } \int yz\,dm = 0,$$

these integrals extending to the entire mass (μ) of the body.

Draw any three lines from this point perpendicular to each other, and relative to these as axes of xyz, let

$$A = \int(y^2+z^2)\,dm, \quad B = \int(x^2+z^2)\,dm, \quad C = \int(x^2+y^2)\,dm,$$

the moments of inertia of the body about the assumed axes of xyz, put also

$$P = \int xy\,dm, \quad Q = \int xz\,dm, \quad R = \int yz\,dm,$$

and draw any right line from the point making angles α, β, γ with these axes, and let r' = radius of gyration about this line, and take on the line a portion $r \propto \frac{1}{r'}$ say $= \frac{k^2}{r'}$, one end of r being the origin (or given point); I say, in the first place, that the locus of the other end of r will be an ellipsoid: for, by Poisson's *Mécanique* (vol. II. p. 55), the moment of inertia about the line, mass $\times r'^2$ or $\mu\frac{k^4}{r^2}$ will be

$$= \sin^2\alpha \int x^2\,dm + \sin^2\beta \int y^2\,dm + \sin^2\gamma \int z^2\,dm - 2\cos\alpha\cos\beta \int xy\,dm$$
$$- 2\cos\alpha\cos\gamma \int xz\,dm - 2\cos\beta\cos\gamma \int yz\,dm,$$

but $$\sin^2\alpha = \cos^2\beta + \cos^2\gamma,$$

since $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1;$$

therefore $$\mu\frac{k^4}{r^2} = \cos^2\alpha\int(y^2+z^2)\,dm + \cos^2\beta\int(x^2+z^2)\,dm$$

$$+ \cos^2\gamma\int(x^2+y^2)\,dm - 2\cos\alpha\cos\beta\int xy\,dm - \&c.;$$

therefore $$= A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - 2P\cos\alpha\cos\beta$$

$$- 2Q\cos\alpha\cos\gamma - 2R\cos\beta\cos\gamma.$$

Now multiply off by r^2, and remembering that $r\cos\alpha$, $r\cos\beta$, $r\cos\gamma$ are the coordinates x, y, z of the end of r, we find

$$\mu k^4 = Ax^2 + By^2 + Cz^2 - 2Pxy - 2Qxz - 2Ryz;$$

therefore the locus of the end of r or point xyz is an ellipsoid, having the given point as centre. Now this ellipsoid remains unaltered, whatever three lines perpendicular to each other be taken as axes of x, y, z, since its radius r is $= \frac{k^2}{r'}$, and r' depends only on the position of the drawn line, as is evident: and since this ellipsoid has three *principal* axes, so that its equation referred to them is of the form

$$\mu k^4 = A'x^2 + B'y^2 + C'z^2,$$

therefore relatively to these *principal* axes P, Q, and R, vanish, *i.e.*

$$\int xy\,dm = 0, \quad \int xz\,dm = 0, \quad \text{and} \quad \int yz\,dm = 0. \quad \text{Q.E.D.}$$

The ellipsoid just determined is evidently Poinsot's Central Ellipsoid, since the moment of inertia (by construction) about any diameter of it varies inversely as the square of the said diameter, and its three *principal* axes are in the directions of the three *principal* axes of the body relatively to this point, as was just proved.

If, on each line drawn from the fixed point, a portion be taken $= r'$ the radius of gyration about this line, then, since $rr' = k^2$, therefore $r' =$ a perpendicular from centre, on a tangent plane to another ellipsoid, called by its discoverer (M'Cullagh) the Ellipsoid of Inertia, which is reciprocal to Poinsot's, and also more intimately connected with the body than Poinsot's, since the radius of gyration itself (and not its reciprocal) about any diameter of this ellipsoid of inertia is given, being equal to the portion of that diameter between the centre and a tangent plane perpendicular to it.

The equation of an ellipsoid being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

the equation of a tangent plane to it, at the point xyz, is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1.$$

Now, at the point where this plane cuts the axis of x, $y'=0$ and $z'=0$, therefore there $xx'=a^2$; let r = the perpendicular let fall from the centre on this tangent plane, and α, β, γ the angles r makes with the three semiaxes, then

$$r=x'\cos\alpha=y'\cos\beta=z'\cos\gamma,$$

and so $rx=a^2\cos\alpha$; in like manner

$$ry=b^2\cos\beta \text{ and } rz=c^2\cos\gamma,$$

and therefore

$$a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma=r^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right), \therefore =r^2,$$

since the equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Now, if $rr'=k^2$, r' being measured from the centre along r, then

$$\frac{k^4}{r'^2}=r^2, \therefore =a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma;$$

multiply off by r'^2, and remembering that

$$r'\cos\alpha, \quad r'\cos\beta, \quad r'\cos\gamma,$$

are the coordinates x, y, z of the end of r', we get

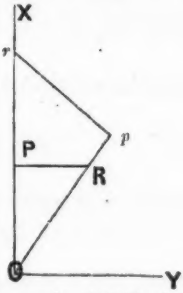
$$a^2x^2+b^2y^2+c^2z^2=k^4,$$

and if $aa'=bb'=cc'=k^2$, this equation gives

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1,$$

and so the locus of the end of r' is another ellipsoid; the two are called reciprocal ellipsoids, since the semidiameter of *either* is proportional to the reciprocal of the distance of the centre from the tangent plane to the other, which is perpendicular to the said semidiameter.

The equation of Poinsot's ellipsoid referred to *any* three rectangular axes drawn from the given point O in the body was found to be



$$Ax^2 + By^2 + Cz^2 - 2Pxy - 2Qxz - 2Ryz = \mu k^4,$$

where A equals the moment of inertia about axis of x, &c., also $P = \int xy\,dm$, &c.; now let the axis of x be OP, perpendicular to tangent plane at R to M'Cullagh's ellipsoid of inertia having its centre at O; let also the axis of y be parallel to PR, so that the point of contact R may be within the angle XOY, the axis of z is, of course, perpendicular to those of x and y; let OP meet Poinsot's ellipsoid in r, and by the property of the reciprocal ellipsoids ORp will be perpendicular to the tangent plane, at r, to Poinsot's ellipsoid, and

$$Or \times OP = OR \times Op = k^2.$$

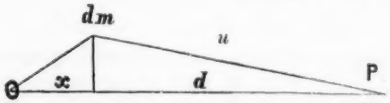
Now, the coordinates of the point r are $y' = 0$, $z' = 0$, and $x' = Or$; and as the tangent plane at r is perpendicular to ORp, and therefore parallel to axis of z, therefore $Q = 0$, else this tangent plane would meet the axis of z, and this being so, the equation of the said tangent plane at r is

$$Axx' - Px'y = \mu k^4, \text{ or } Ax - Py = \frac{\mu k^4}{x'},$$

which shows that tangent of rOp or $\frac{PR}{PO}$ is $= -\frac{P}{A}$, and as $A = \mu \times OP^2$, therefore

$$P \text{ or } \int xy\,dm = -\mu \times OP \times PR, \therefore = -\mu \times 2\Delta OPR.$$

Now, let O be the centre of gravity of a body of any



shape, attracting a very remote point P, take $OP = d$ as axis of x, let dm be a particle of the body, and u its distance from P, then

$$X = \Sigma \frac{dm}{u^2} \times \frac{d - x}{u}, \quad Y = \Sigma \frac{dm}{u^2} \times \frac{y}{u}, \quad \text{and } Z = \Sigma \frac{dm}{u^2} \times \frac{z}{u},$$

and as $u^2 = d^2 - 2dx + x^2 + y^2 + z^2$;

therefore $u^{-3} = d^{-3}\left(1 + \frac{3x}{d} + \frac{12x^2 - 3y^2 - 3z^2}{2d^2}\right)$ nearly,

and this multiplied by $d - x = d\left(1 - \frac{x}{d}\right)$ gives

$$d^{-2}\left(1 + \frac{2x}{d} + \frac{6x^2 - 3y^2 - 3z^2}{2d^2}\right),$$

and as O is centre of gravity, therefore

$$\Sigma x\,dm = 0 = \Sigma y\,dm = \Sigma z\,dm;$$

therefore $X = d^{-2}\Sigma dm\left(1 + \frac{2x}{d} + \frac{6x^2 - 3y^2 - 3z^2}{2d^2}\right)$,

therefore $= \frac{\mu}{d^2} + \frac{3}{2d^4}\Sigma(2x^2 - y^2 - z^2)\,dm.$

Now $\Sigma(2x^2 - y^2 - z^2)dm = \Sigma(2x^2 + 2y^2 + 2z^2)dm - \Sigma(3y^2 + 3z^2)dm,$

and as $x^2 + y^2 + z^2$ equals the square of the distance of dm from O, therefore $= x'^2 + y'^2 + z'^2$, x', y', z' being the coordinates of dm referred to the *principal* axes at O; and as

$$\Sigma(y^2 + z^2)\,dm = M,$$

the moment of inertia about OP and

$$A = \Sigma(y'^2 + z'^2)\,dm, \quad B = \Sigma(x'^2 + z'^2)\,dm, \quad C = \Sigma(x'^2 + y'^2)\,dm;$$

therefore $A + B + C = \Sigma(2x'^2 + 2y'^2 + 2z'^2)\,dm,$

therefore $= \Sigma(2x^2 + 2y^2 + 2z^2)\,dm,$

and so $$X = \frac{\mu}{d^2} + \frac{3}{2d^4}(A + B + C - 3M).$$

Now, if M'Cullagh's ellipsoid of inertia be made, having O as its centre, its *principal* axes a, b, c coinciding in direction with the *principal* axes of the body at O, and

$$A = \mu a^2, \quad B = \mu b^2, \quad C = \mu c^2,$$

then, if a tangent plane to this ellipsoid, perpendicular to OP at P', touch it in R, and if the axis of y be taken parallel to $P'R$, and in the same sense (or direction) the axis of z being perpendicular to those of x and y, with such axes we proved above, that

$$\Sigma xz\,dm = 0\ (= Q) \text{ and } \Sigma xy\,dm = -\mu \times OP' \times P'R;$$

hence then, $Z = \Sigma\frac{z\,dm}{u^3} = d^{-3}\Sigma dm\left(z + \frac{3xz}{d}\right)$, $\therefore = 0,$

and $Y = \Sigma \frac{ydm}{u^3} = d^{-3}\Sigma dm \left(y + \frac{3xy}{d}\right), \therefore = \frac{3}{a^4} \Sigma xy\, dm$;

therefore (by what is already proved)

$$= -\frac{3\mu}{d^4} \times OP' \times P'R,$$

the negative sign indicating that the attraction of the body on P, in a direction perpendicular to OP, is parallel to RP' and not to $P'R$.

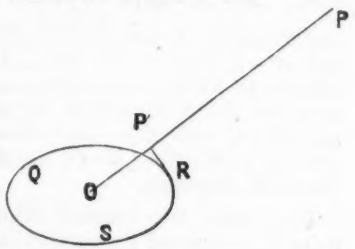
If two confocal ellipsoids attract an external point, their two resultants are coincident in direction, and proportional to their masses.

A being the attracting ellipsoid, whose foci are given, and P the external attracted point, let B be an ellipsoid, confocal to A, and passing through P, and let P' be the point on A *corresponding* to P on B; let also abc, $a'b'c'$ be the semiaxes of A and B, $a'b'c'$ being fixed, let $x'y'z'$ be the coordinates of P and therefore given, and let xyz be the coordinates of P' on the variable ellipsoid A, whose foci are given. Now the attraction of B on P' parallel to $a \propto x$,* therefore $\propto a$ since $\frac{x}{a} = \frac{x'}{a'}$, therefore equal a constant; but by Ivory's theorem, attraction of A on P, parallel to a, is to foregoing attraction :: $bc : b'c'$, and so attraction of A on P, parallel to $a \propto abc$, therefore varies as mass of A, and in like manner the components of A's attraction on P, parallel to b and c, are varies as mass of A; thus, when A is changed into another ellipsoid, confocal to it, the three components of its attraction on P, parallel to the semiaxes a, b, c, are all changed in the ratio of the mass of A, therefore the direction of the resultant remains invariable, but its quantity is altered in proportion to the mass of A.

We found

$$\frac{3\mu}{d^4} \times OP' \times P'R$$

above for the attraction of a body of any shape on a very distant point P, in a direction perpendicular to OP, μ being the mass of the attracting body, O its centre of gravity,



* I shall give an elementary geometrical proof of this hereafter.

$OP = d$ and RSQ the ellipsoid of inertia (M'Cullagh's, and not Poinsot's) relative to the centre of gravity O, and R the point of contact of a tangent plane to this ellipsoid which cuts OP at right angles in P'. The said attraction perpendicular to OP was also proved parallel to RP' (not to $P'R$): let the semiaxes of the ellipsoid be abc, and let $OP' = r$ make angles $\alpha\beta\gamma$ with abc, and let the coordinates of R be xyz; then the said attraction of force acting on P parallel to RP', being resolved into three components parallel to a, b, c respectively, the component parallel to a will obviously be

$$\frac{3\mu}{d^4} r(r\cos\alpha - x),$$

since $r\cos\alpha - x$ is the projection of RP' on a. But we proved before that $rx = a^2\cos\alpha$, therefore the component parallel to a

$$= \frac{3\mu}{d^4}\cos\alpha\,(r^2 - a^2), \quad \therefore = \frac{3\cos\alpha}{d^4}(M - A),$$

since by the property of the ellipsoid of inertia $\mu \times OP'^2 = M$, the moment of inertia about OP, and $A = \mu a^2$; similarly, of course, the components parallel to b and c are

$$\frac{3\cos\beta}{d^4}(M - B), \text{ and } \frac{3\cos\gamma}{d^4}(M - C).$$

Now

$$M - A = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - A(\cos^2\alpha + \cos^2\beta + \cos^2\gamma),$$
$$\therefore = (B - A)\cos^2\beta + (C - A)\cos^2\gamma;$$

therefore, in case of an attracting ellipsoid, where

$$A = \tfrac{1}{5}\mu(b^2 + c^2), \quad B = \tfrac{1}{5}\mu(a^2 + c^2), \quad \text{and} \quad C = \tfrac{1}{5}\mu(a^2 + b^2),$$

then $$M - A = \tfrac{1}{5}\mu(a^2 - b^2)\cos^2\beta + \tfrac{1}{5}\mu(a^2 - c^2)\cos^2\gamma;$$

and therefore, in case of two confocal ellipsoids, where $a^2 - b^2$ and $a^2 - c^2$ will be the same, $M - A \propto \mu$; the line OP, relative to which M is calculated, being also supposed the same for the two confocal ellipsoids. Hence also $M - B$ and $M - C$ will $\propto \mu$, and therefore we shall also have $(M - A) + (M - B) + (M - C)$ or $A + B + C - 3M \propto$ mass for two such ellipsoids.

Now, when two confocal ellipsoids attract *any* external point P, we proved their attractions are as their masses, and coincident in direction; and supposing one of them given, and large, but of small eccentricity, and conceive the other

to be confocal to it, and very small, so that P may be a remote point relatively to it; then the components parallel to a, b, c of the small one's attraction on P, perpendicularly to the straight line PO, joining P to the centre O, being

$$\frac{3\cos\alpha}{d^4}(M'-A'),\ \&c,$$

therefore the analogous components of the large one's attraction will be $\frac{3\cos\alpha}{d^4}(M-A)$, for the $M-A$ of such ellipsoids was just proved to vary as their masses: and since the small ellipsoid's attraction on P (along $PO=d$), which may be considered a remote point relatively to this small one, is

$$\frac{\mu'}{d^2}+\frac{3}{2d^4}(A'+B'+C'-3M'),$$

and since the attractions of the two ellipsoids are in the same direction, and as their masses, therefore the attraction of the large given ellipsoid on the external point P, along PO, is

$$\frac{\mu}{d^2}+\frac{3}{2d^4}(A+B+C-3M),$$

since it was proved above that

$$A+B+C-3M : A'+B'+C'-3M' :: \text{mass}\,\mu : \text{mass}\,\mu'.$$

Now, since the preceding expressions for the attraction along PO, and for the components parallel to a, b, c, are linear functions of μ, A, B, C, and hold true for *any* ellipsoid of small excentricities, it follows therefore that they hold true also for an ellipsoid which is not homogeneous, but composed of concentric ellipsoidal strata, having the same *principal* axes and variable but small excentricities.

If the ellipsoid become a spheroid, so that $c=b=a(1+e)$, then the central attraction on P, along PO,

$$\frac{\mu}{d^2}+\frac{3}{2d^4}(A+B+C-3M)$$

on account of $C=B$ becomes

$$\frac{\mu}{d^2}+\frac{3}{2d^4}\{A+2B-3(B\cos^2\lambda+A\sin^2\lambda)\}$$

$$=\frac{\mu}{d^2}+\frac{3}{2d^4}(A-B)(1-3\sin^2\lambda)=\frac{\mu}{d^2}+T(1-3\sin^2\lambda),$$

where $T = \frac{3}{2d^4}(A-B)$ and $d = OP$, the distance of the attracted point P (now supposed on the surface of the spheroid) from the centre, and λ equals the angle d makes with the equator, *i.e.* equals the latitude of P.

The component parallel to a of the force perpendicular to OP, was $= \frac{3\cos\alpha}{d^4}(M-A)$; and since

$$M = B\cos^2\lambda + A\sin^2\lambda, \text{ and } \therefore M - A = (B-A)\cos^2\lambda,$$

therefore the said component

$$= -\frac{3}{d^4}(A-B)\sin\lambda\cos^2\lambda;$$

and in like manner the component parallel to b is

$$= \frac{3}{d^4}(A-B)\cos\lambda\sin^2\lambda:$$

therefore the whole force or attraction on P perpendicular to PO, being the square root of the sum of the squares of these components, is

$$= \frac{3}{d^4}(A-B)\sin\lambda\cos\lambda, \ \therefore = T\sin 2\lambda,$$

and tends to urge P towards the equator (on account of the negative sign of the component parallel to a).

Now the equation of a meridianal section of the exterior surface is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1;$$

and as $x = d\cos\lambda$ and $y = d\sin\lambda$,

therefore $$\frac{\cos^2\lambda}{a^2(1+e)^2} + \frac{\sin^2\lambda}{a^2} = \frac{1}{d^2};$$

therefore $\frac{a^2}{d^2} = 1 - 2e\cos^2\lambda$, and so $d = a(1 + e\cos^2\lambda)$.

Let θ be the angle between $d = OP$ and the normal at P; then

$$\tan\theta = \frac{-\text{differential } d}{d \times \text{differential } \lambda}, \ \therefore = e\sin 2\lambda \text{ nearly},$$

and $\sin\theta = \cos\theta\tan\theta$, $\therefore = \tan\theta$ nearly, $\therefore = e\sin 2\lambda$ nearly, θ being small, and therefore $\cos\theta = 1$ nearly. Now, that the point P may be at rest, it is necessary that the tangential component of the central force acting along PO should be equal to the sum of the tangential components of the cen-

trifugal force (acting on P parallel to b), and of the force perpendicular to PO; which condition gives the equation

$$\left\{\frac{\mu}{d^2} + T(1 - 3\sin^2\lambda)\right\}\sin\theta = T\sin 2\lambda\cos\theta + g\phi\cos\lambda\sin\lambda,$$

where g is the mean force of gravity at the surface, and $g\phi$ is the centrifugal force at the equator; but since T is small compared to $\frac{\mu}{d^2}$ or g, and since $\cos\theta = 1$, and $\sin\theta = e\sin 2\lambda$ (rejecting e^2, e^3, &c.), the preceding equation becomes therefore

$$ge\sin 2\lambda = T\sin 2\lambda + \tfrac{1}{2}g\phi\sin 2\lambda,$$

or therefore

$$e = \frac{T}{g} + \frac{\phi}{2}.$$

Now, the formula $\frac{\mu}{d^2} + T(1 - 3\sin^2\lambda)$ gives $\frac{\mu}{a^2} - 2T$ for the polar gravity, and $\frac{\mu}{a^2(1+e)^2} + T - g\phi$ for the equatorial gravity, their difference is $\frac{\mu}{a^3} \times 2e + g\phi - 3T$: if this difference, divided by the mean gravity g, be denoted by n, then

$$n = 2e + \phi - \frac{3T}{g} \text{ nearly};$$

and eliminating $\frac{T}{g}$ from this and the preceding equation $e = \frac{T}{g} + \frac{\phi}{2}$, we get $n + e = \frac{5\phi}{2}$, which is Clairaut's Theorem. If the earth be homogeneous, then $n = e$, $\therefore = \frac{5\phi}{4}$. See the *Principia*, Prop. 20, Book III.

THE EQUATIONS OF SURFACES CONSIDERED AS SUMS OF HOMOGENEOUS FUNCTIONS.

By the Rev. Robert Carmichael,
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The theorems contained in the following paper are principally generalizations of others long familiar. Although, so far as I am aware, they are, with the exception of that

given in the fifth article, new; yet, from slight acquaintance with the theory of determinants, I am unable to say whether any of them have been previously published.* The paper then is mainly intended for the student, not for the advanced mathematician. It may, however, serve to systematize many results hitherto regarded as isolated, and to exhibit the convenience of adopting a more symmetrical notation in mathematical works than has been hitherto employed—unless in detached cases. I may observe that the principal results were obtained in the month of December 1851, and communicated to the Dublin University Philosophical Society in the early part of the year 1852.

1. It is easily seen that if we express any rational and integer function of x, y, z, as a sum of a number of homogeneous functions,

$$U = u_n + u_{n-1} + \ldots + u_1 + u_0;$$

then

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\right) U = nu_n + (n-1)\,u_{n-1} + \ldots + u_1.$$

Hence if

$$U = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1.)$$

be the general equation of the surface of the n^{th} degree, and α, β, γ the coordinates of any point

$$\left(x\frac{d}{dx} + y\frac{d}{dy} + z\frac{d}{dz}\right) U = -\left(u_{n-1} + 2u_{n-2} + \ldots + nu_0\right),$$

and consequently the equation of the polar surface of (α, β, γ) with respect to $U = 0$ is

$$\alpha\frac{dU}{dx} + \beta\frac{dU}{dy} + \gamma\frac{dU}{dz} + u_{n-1} + 2u_{n-2} + \ldots + nu_0 = 0.$$

The equation of the polar surface to the origin, then, is

$$u_{n-1} + 2u_{n-2} + 3u_{n-3} + \ldots + nu_0 = 0,$$

which may be easily verified in the case in which $U = 0$ is of the second degree.

2. If however (α, β, γ) be taken as the current coordinates of the plane tangent at the point (x, y, z), it appears that the

* Since the above was written, I have been informed by Mr. Salmon that some of the theorems in this paper are included under the head of "contravariants," as treated by Mr. Sylvester. The point of view, however, from which they are here regarded is quite distinct, and the analytical discussion, as far as possible, elementary.

equation of this plane is, generally,

$$\alpha \frac{dU}{dx} + \beta \frac{dU}{dy} + \gamma \frac{dU}{dz} + u_{n-1} + 2u_{n-2} + \ldots + nu_0 = 0,$$

and that the general expression for the perpendicular from the origin on the tangent plane is

$$P = -\frac{u_{n-1} + 2u_{n-2} + \ldots + nu_0}{\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 + \left(\frac{dU}{dz}\right)^2\right\}^{\frac{1}{2}}}.$$

Hence the theorem—"Given a surface of the n^{th} degree, the points thereon for which the perpendicular on tangent plane is constant (k), lie on another surface of the degree $2(n-1)$, whose equation is

$$k^2\left\{\left(\frac{dU}{dx}\right)^2 + \left(\frac{dU}{dy}\right)^2 + \left(\frac{dU}{dz}\right)^2\right\} = (u_{n-1} + 2u_{n-2} + \ldots + nu_0)^2."$$

The general expression admits of instant verification in the simple case

$$U = u_n - c = 0.$$

It has been remarked to me by Mr. Spottiswoode that the polar surface to the origin passes through all the singular points of the surface $U = 0$ at which the perpendicular from the origin on the tangent plane is not infinite. In fact, at a singular point

$$\frac{dU}{dx} = 0, \quad \frac{dU}{dy} = 0, \quad \frac{dU}{dz} = 0,$$

whence $P = \infty$,

unless $u_{n-1} + 2u_{n-2} + 3u_{n-3} + \ldots + nu_0 = 0$

be satisfied.

3. Let now the point (α, β, γ) be supposed capable of motion on the surface of the m^{th} degree

$$V = v_m + v_{m-1} + \ldots + v_1 + v_0 = 0 \ldots\ldots\ldots\ldots (II),$$

and let it assume various consecutive positions on this surface. The corresponding successive polars, taken with respect to $U = 0$, will by their intersections generate a third surface, whose relation to (II.) is commonly expressed by the distinctive appellation of Reciprocal Polar, for the case in which (I.) is of the second order.

To find the equation of this third surface, differentiating (II.) and the general equation of the polar surface with

respect to α, β, γ, we get

$$\frac{dV}{d\alpha}\,d\alpha + \frac{dV}{d\beta}\,d\beta + \frac{dV}{d\gamma}\,d\gamma = 0,$$

$$\frac{dU}{dx}\,d\alpha + \frac{dU}{dy}\,d\beta + \frac{dU}{dz}\,d\gamma = 0.$$

Multiplying the latter equation by the indeterminate quantity λ, adding, and putting the coefficients of $d\alpha$, $d\beta$, $d\gamma$, respectively, equal to zero, there results the system

$$\left.\begin{aligned} \frac{dV}{d\alpha} + \lambda\frac{dU}{dx} &= 0 \\ \frac{dV}{d\beta} + \lambda\frac{dU}{dy} &= 0 \\ \frac{dV}{d\gamma} + \lambda\frac{dU}{dz} &= 0 \end{aligned}\right\};$$

and between this, the equation of the polar surface, and (11), we have to eliminate α, β, γ, and λ.

To accomplish this, we multiply the three equations of the last system by α, β, γ, respectively, and remembering that

$$\alpha\frac{dV}{d\alpha} + \beta\frac{dV}{d\beta} + \gamma\frac{dV}{d\gamma} = -\{v_{m-1} + 2v_{m-2} + \ldots + mv_0\},$$

$$\alpha\frac{dU}{dx} + \beta\frac{dU}{dy} + \gamma\frac{dU}{dz} = -\{u_{n-1} + 2u_{n-2} + \ldots + nu_0\},$$

we find that

$$\lambda = -\frac{v_{m-1} + 2v_{m-2} + \ldots + mv_0}{u_{n-1} + 2u_{n-2} + \ldots + nu_0} = -\frac{(V)}{(U)}.$$

Thus it remains for us to eliminate α, β, γ between the four equations

$$V = 0,$$

and

$$\left.\begin{aligned} \frac{1}{(V)}\cdot\frac{dV}{d\alpha} &= \frac{1}{(U)}\cdot\frac{dU}{dx} \\ \frac{1}{(V)}\cdot\frac{dV}{d\beta} &= \frac{1}{(U)}\cdot\frac{dU}{dy} \\ \frac{1}{(V)}\cdot\frac{dV}{d\gamma} &= \frac{1}{(U)}\cdot\frac{dU}{dz} \end{aligned}\right\},$$

where the left-hand members contain only α, β, and γ, and the right-hand only x, y, and z.

Such an elimination, in the present state of analysis, is I believe impossible, and the general question therefore insoluble. Thus the only general representation of the envelope of the successive polar surfaces is the *system* of four equations last mentioned.

4. Upon communicating the above result to Mr. Spottiswoode, it was observed by him that the three last equations may be written in a new form, possibly leading to interesting consequences, and I am indebted to the Rev. R. Townsend for a valuable modification of his suggestions.

If we remember that the point (α, β, γ) lies on the surface $V = 0$, it is obvious that, P being the perpendicular from the origin on the tangent-plane at this point, and l, m, n the angles made by it with the coordinate axes, we may write those three last equations in the form

$$\left.\begin{aligned} -\frac{\cos l}{P} &= \frac{1}{(U)} \cdot \frac{dU}{dx} \\ -\frac{\cos m}{P} &= \frac{1}{(U)} \cdot \frac{dU}{dy} \\ -\frac{\cos n}{P} &= \frac{1}{(U)} \cdot \frac{dU}{dz} \end{aligned}\right\}.$$

It is evident that the right-hand members of the system do not admit of a modification similar to that which we have employed on the left hand, since the point (x, y, z) is *not* necessarily on the surface $U = 0$.

5. In one case, the general question of the envelope of the successive polars not only admits of solution, but the resultant equation of the envelope appears to possess both elegance and utility. It is that in which (11) assumes the symmetrical form

$$\frac{\alpha^m}{a^m} + \frac{\beta^m}{b^m} + \frac{\gamma^m}{c^m} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (11)',$$

while (1) still retains all its generality. (The ordinary reciprocal of (11′) was given many years ago by Mr. Salmon.)

The three last equations of the third article, in this case, become

$$\frac{\alpha^{m-1}}{a^m} + \frac{1}{(U)} \cdot \frac{dU}{dx} = 0,$$

$$\frac{\beta^{m-1}}{b^m} + \frac{1}{(U)} \cdot \frac{dU}{dy} = 0,$$

$$\frac{\gamma^{m-1}}{c^m} + \frac{1}{(U)} \cdot \frac{dU}{dz} = 0,$$

and eliminating α, β, γ between these equations and (11), we get the equation of the envelope required, in the symmetrical form

$$\left(a\frac{dU}{dx}\right)^{\frac{m}{m-1}} + \left(b\frac{dU}{dy}\right)^{\frac{m}{m-1}} + \left(c\frac{dU}{dz}\right)^{\frac{m}{m-1}} = [-(U)]^{\frac{m}{m-1}},$$

where $$(U) = u_{n-1} + 2u_{n-2} + 3u_{n-3} + \ldots + nu_0.$$

The discussion of some particular cases will be found to lead to interesting results.

(1) When $m = 2$, or when the pole is confined to a central surface of the second degree, then will the degree of the envelope of the successive polars with respect to a surface of the n^{th} degree be, in general,

$$2(n-1).$$

(2) When moreover the surface, with respect to which the polar is taken, is also of the second degree and central, the envelope, or now *Reciprocal Polar*, to

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 1$$

will be a third surface of the second degree and its equation takes the symmetrical form

$$a^2\left(\frac{dU}{dx}\right)^2 + b^2\left(\frac{dU}{dy}\right)^2 + c^2\left(\frac{dU}{dz}\right)^2 = (u_1 + u_0)^2.$$

(3) When the pole is confined to a central *curve* of the *second* degree and the polar taken with respect to any curve of the *third*, the envelope of the successive polars is a curve of the *fourth* degree, which is symmetrically represented by the equation

$$a^2\left(\frac{dU}{dx}\right)^2 + b^2\left(\frac{dU}{dy}\right)^2 = (u_2 + 2u_1 + 3u_0)^2,$$

a result which seems susceptible of elegant application to the theory of curves of the third degree.

NOTE ON THE TRANSFORMATION OF A TRIGONOMETRICAL EXPRESSION.

By Arthur Cayley.

The differential equation

$$\frac{dx}{(a+x)\sqrt{(c+x)}} + \frac{dy}{(a+y)\sqrt{(c+y)}} + \frac{dz}{(a+z)\sqrt{(c+z)}} = 0,$$

integrated so as to be satisfied when the variables are simultaneously infinite, gives by direct integration

$$\tan^{-1}\sqrt{\left(\frac{a-c}{c+x}\right)} + \tan^{-1}\sqrt{\left(\frac{a-c}{c+y}\right)} + \tan^{-1}\sqrt{\left(\frac{a-c}{c+z}\right)} = 0.$$

And, by Abel's theorem,

$$\begin{vmatrix} 1, & x, & (a+x)\sqrt{(c+x)} \\ 1, & y, & (a+y)\sqrt{(c+y)} \\ 1, & z, & (a+z)\sqrt{(c+z)} \end{vmatrix} = 0.$$

To shew *a posteriori* the equivalence of these two equations, I represent the determinant by the symbol $\square$, and expressing it in the form

$$\square = \begin{vmatrix} 1, & a+x, & (a+x)\sqrt{(c+x)} \\ \vdots & & \end{vmatrix}$$

I write for the moment $\xi = \sqrt{\left(\frac{a-c}{c+x}\right)}$ &c., this gives

$$\square = \begin{vmatrix} 1, & (a-c)\left(1+\frac{1}{\xi^2}\right), & (a-c)^{\frac{3}{2}}\left(\frac{1}{\xi}+\frac{1}{\xi^3}\right) \\ \vdots & & \end{vmatrix}$$

$$= \frac{(a-c)^{\frac{3}{2}}}{\xi^3\eta^3\zeta^3} \begin{vmatrix} \xi^3, & \xi^2+\xi, & \xi^2+1 \\ \vdots & & \end{vmatrix}$$

$$= \frac{(a-c)^{\frac{3}{2}}}{\xi^3\eta^3\zeta^3} \begin{vmatrix} \xi^3, & \xi, & \xi^2+1 \\ \vdots & & \end{vmatrix}$$

$$= -\frac{(a-c)^{\frac{3}{2}}}{\xi^3\eta^3\zeta^3} \left\{ \begin{vmatrix} 1, & \xi, & \xi^3 \\ \vdots & & \end{vmatrix} - \xi\eta\zeta \begin{vmatrix} 1, & \xi, & \xi^2 \\ \vdots & & \end{vmatrix} \right\}$$

$$= -\frac{(a-c)^{\frac{3}{2}}}{\xi^3\eta^3\zeta^3} (\xi+\eta+\zeta-\xi\eta\zeta) \begin{vmatrix} 1, & \xi, & \xi^2 \\ \vdots & & \end{vmatrix}$$

Or, replacing ξ, η, ζ by their values, we have identically

$$\begin{vmatrix} 1, & x, & (a+x)\sqrt{(c+x)} \\ 1, & y, & (a+y)\sqrt{(c+y)} \\ 1, & z, & (a+z)\sqrt{(c+z)} \end{vmatrix} = -\frac{(c+x)^{\frac{3}{2}}(c+y)^{\frac{3}{2}}(c+z)^{\frac{3}{2}}}{(a-c)^2} \left\{ \sqrt{\frac{a-c}{c+x}} + \sqrt{\frac{a-c}{c+y}} + \sqrt{\frac{a-c}{c+z}} - \sqrt{\frac{a-c}{c+x}}\sqrt{\frac{a-c}{c+y}}\sqrt{\frac{a-c}{c+z}} \right\} \begin{vmatrix} 1, & \sqrt{\frac{a-c}{c+x}}, & \frac{a-c}{c+x} \\ 1, & \sqrt{\frac{a-c}{c+y}}, & \frac{a-c}{c+y} \\ 1, & \sqrt{\frac{a-c}{c+z}}, & \frac{a-c}{c+z} \end{vmatrix}$$

And the equation

$$\sqrt{\frac{a-c}{c+x}} + \sqrt{\frac{a-c}{c+y}} + \sqrt{\frac{a-c}{c+z}} - \sqrt{\frac{a-c}{c+x}}\sqrt{\frac{a-c}{c+y}}\sqrt{\frac{a-c}{c+z}} = 0$$

is of course equivalent to the trigonometrical equation

$$\tan^{-1}\sqrt{\frac{a-c}{c+x}} + \tan^{-1}\sqrt{\frac{a-c}{c+y}} + \tan^{-1}\sqrt{\frac{a-c}{c+z}} = 0,$$

which shews the equivalence of the two equations in question.

2, *Stone Buildings, Lincoln's Inn,*
November 3, 1853.

REMARQUES SUR UN MEMOIRE DE M. CAYLEY RELATIF AUX DETERMINANTS GAUCHES.

Par M. HERMITE.

Mr. CAYLEY a nommé système gauche symétrique, un système de n^2 quantités représentées par $\lambda_{r,s}$ en attribuant aux indices toutes les valeurs entières depuis 1 jusqu' à n, lorsqu' on a la condition générale

$$\lambda_{r,s} = -\lambda_{s,r},$$

d' où resulte, $\lambda_{r,r} = 0.$

De pareils systèmes jouissent de propriétés importantes qui jouent au grand rôle dans les diverses circonstances analytiques ou ils se présentent, et Mr. Cayley en a fait lui même un nouvel usage pour la solution de cette question—

Obtenir toutes les transformations d' une forme quadratique en elle-même lorsque cette forme est une somme de carrés.

Je me propose de donner içi des formules analogues à celles de M. Cayley, pour la transformation en elle-même d'une forme quadratique quelconque. Le problème peut-être posé ainsi $f(x_1, x_2, \ldots x_n)$ designant la forme quadratique proposée ; trouver l'expression la plus générale des quantités $X_1, X_2, \ldots X_n$, qui donnent

$$f(X_1, X_2, \ldots X_n) = f(x_1, x_2, \ldots x_n).$$

Pour cela, j'imagine que les quantités X et x soient exprimées par des indéterminées auxiliaires ξ, de sorte qu' on ait en general

$$X_r + x_r = 2\xi_r,$$

et sous cette condition on va voir qu' il est très facile d' obtenir l' expression générale de X et x en ξ. On a en effet

$$X_r = 2\xi_r - x_r,$$

donc $f(X_1, X_2, \ldots) = f(2\xi_1 - x_1,\ 2\xi_2 - x_2, \ldots)$.....(1),

ou en developpant le second membre

$$f(X_1, X_2, \ldots) = 4f(\xi_1, \xi_2, \ldots) - 2\left\{x_1 \frac{df}{d\xi_1} + x_2 \frac{df}{dx_2} + \ldots\right\}$$
$$+ f(x_1, x_2, \ldots).....(2).$$

Donc par la condition supposée

$$f(X_1, X_2, \ldots) = f(x_1, x_2, \ldots),$$

cette équation se reduit à

$$x_1 \frac{df}{d\xi_1} + x_2 \frac{df}{d\xi_2} + \ldots = 2f \ldots\ldots\ldots\ldots\ldots (3).$$

Or la manière le plus générale de la vérifier en exprimant les quantités x en ξ, sera de faire

$$x_r = \xi_r + \tfrac{1}{2}\sum_1^n{}_s \lambda_{r,s} \frac{df}{d\xi_s} \ldots\ldots\ldots\ldots\ldots (4),$$

les indéterminées λ étant assujetées à la condition

$$\lambda_{r,s} = -\lambda_{s,r}.$$

Ou en conclut

$$X_r = 2\xi_r - x_r = \xi_r - \tfrac{1}{2}\sum_1^n{}_s \lambda_{r,s} \frac{df}{d\xi_s},$$

et il est facile de reconnaitre *a posteriori* que ces expressions de X et x en ξ, donnent bien

$$f(X_1, X_2, \ldots) = f(x_1, x_2, \ldots) \ldots\ldots\ldots\ldots (5).$$

Reprennant en effet l'équation (1) et l'équation (2), on verra par l'équation (3), équation satisfaite d'elle-même, qu'on retombe précisément sur l'équation (5) qui était à vérifier. Donc enfin les expressions cherchées de X et x, qui donnent la transformation en elle-même d'une forme quelconque, s'obtiendront en résolvant par rapport aux quantités ξ les équations (4), et substituant les valeurs en x, qu'on aura trouvées de la sorte, dans les formules

$$X_r = 2\xi_r - x_r.$$

Considerons pour application les former binaires

$$f = ax^2 + 2bxy + cy^2,$$

ou nous mettons x et y, au lieu de x_1 et x_2; nous aurons successivement

$$X + x = 2\xi,$$

$$Y + y = 2\eta,$$

et
$$x = \xi + \lambda(b\xi + c\eta) = \xi(1 + \lambda b) + \lambda c\eta,$$

$$y = \eta - \lambda(a\xi + b\eta) = -\lambda a\xi + (1 - \lambda b)\eta.$$

D'ou en résolvant

$$\xi = \frac{(1 - \lambda b)\, x - \lambda c y}{1 - \lambda^2(b^2 - ac)},$$

$$\eta = \frac{\lambda a x + (1 + \lambda b)\, y}{1 - \lambda^2(b^2 - ac)};$$

Soit pour abréger

$$b^2 - ac = D,$$

on trouvera

$$X = 2\xi - x = \frac{(1 - 2\lambda b + \lambda^2 D)\, x - 2\lambda c y}{1 - \lambda^2 D},$$

$$Y = 2\eta - y = \frac{2\lambda a x + (1 + 2\lambda b + \lambda^2 D)\, y}{1 - \lambda^2 D}.$$

Or ces formules en posant,

$$t = \frac{1 + \lambda^2 D}{1 - \lambda^2 D},$$

$$u = \frac{2\lambda}{1 - \lambda^2 D};$$

ce qui donne

$$t^2 - Du^2 = 1,$$

deviendront

$$X = x(t - bu) - cuy,$$
$$Y = xau + (t + bu)\, y.$$

C'est la forme analytique obtenue par M. Gauss pour la question arithmétique où l'on veut que les coefficients de la substitution soient des nombres entiers.

Enfin si l'on fait l'application de la même méthode à une forme quadratique d'un nombre quelconque d'indéterminées dans le cas ou elle est une somme de carrés, on se trouvera immédiatement les résultats que M. Cayley a obtenus dans son beau mémoire, et je m'empresse de dire que je dois à l'étude de ce mémoire, l'analyse que je viens d'expresser en peu de mots. J'ajouterai cependant encore les théorèmes suivants qui servent de lemmes à une recherche arithmétique importante.

I. Ayant ramené à une somme de carrés de fonctions linéaires une forme quadratique quelconque, de sorte qu'on ait *p.ex.*

$$f = A^2 + B^2 + C^2 + \&c.$$

Si l'on désigne par $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, &c., ce que deviennent respectivement A, B, C, ... lorsqu'on fait dans f, une substitution quelconque qui la change en elle-même, on aura évidemment

$$\mathfrak{A} = \alpha A + \beta B + \gamma C + \ldots$$
$$\mathfrak{B} = \alpha' A + \beta' B + \gamma' C + \ldots$$
$$\mathfrak{C} = \alpha'' A + \beta'' B + \gamma'' C + \ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

les quantités α, β, γ, &c., etant des constantes convenablement choisies. Cela posè, à une substitution qui change f en elle-même, ou pourra toujours faire correspondre une telle représentation de f par la forme $A^2 + B^2 + C^2 + \ldots$, que l'expression

$$A\mathfrak{A} + B\mathfrak{B} + C\mathfrak{C} + \text{&c.},$$

ne contienne aucun des rectangles AB, AC, &c.

Pour donner une application de ce théorème, nous allons considérer le cas des formes quadratiques ternaires

$$f = A^2 + B^2 + C^2.$$

Alors les constantes α, β, γ, &c., devant être telles que

$$\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2 = A^2 + B^2 + C^2,$$

auront d'après M. Cayley, les valeurs suivantes,

$$\kappa\alpha = 1 + \lambda^2 - \mu^2 - \nu^2, \quad \kappa\alpha' = 2(\lambda\mu - \nu), \quad \kappa\alpha'' = 2(\lambda\nu + \mu),$$

$$\kappa\beta = 2(\mu\lambda + \nu), \quad \kappa\beta' = 1 - \lambda^2 + \mu^2 - \nu^2, \quad \kappa\beta'' = 2(\mu\nu - \lambda),$$

$$\kappa\gamma = 2(\nu\lambda - \mu), \quad \kappa\gamma' = 2(\nu\mu + \lambda), \quad \kappa\gamma'' = 1 - \lambda^2 - \mu^2 + \nu^2,$$

où

$$\kappa = 1 + \lambda^2 + \mu^2 + \nu^2,$$

et à toute substitution S qui change f en elle-même, on pourra toujours faire correspondre un système de fonctions linéaires A, B, C, jouissant de la propriété, qu'en devenant respectivement $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, lorsqu'on effectue la substitution S, ou aura les relations

$$\alpha' + \beta = 0, \quad \alpha'' + \gamma = 0, \quad \beta'' + \gamma' = 0.$$

De-là se tire la conclusion que l'une des quantités λ, μ, ν est zero. On peut donc faire par exemple

$$\mathfrak{A}^2 = A^2,$$

$$\mathfrak{B}^2 + \mathfrak{C}^2 = B^2 + C^2;$$

ou bien

$$\mathfrak{A} = \pm A,$$

$$\mathfrak{B} = B\cos\theta + C\sin\theta,$$

$$\mathfrak{C} = B\sin\theta - C\cos\theta.$$

De là ces théorèmes:

II. Soit

$$X = px + p'y + p''z,$$

$$Y = qx + q'y + q''z,$$

$$Z = rx + r'y + r''z.$$

une substitution qui change en elle-même une forme ternaire quelconque; l'une des racines λ de l'équation

$$\Lambda = \begin{vmatrix} p-\lambda & p' & p \\ q & q'-\lambda & q'' \\ r & r' & r''-\lambda \end{vmatrix} = 0;$$

sera égale à ± 1, et les deux autres seront réciproques.

III. Il existe une infinité de formes ternaires différentes de f que la substitution ci-dessus change en elles-mêmes, ces formes seront toutes données par l'expression

$$F = kA^2 + l(B^2 + C^2),$$

k et l étant des constantes arbitraires. Cependant le cas des racines égales dans l'équation $\Lambda = 0$, doit être traité à part et exige une discussion spéciale, que nous laisserons faire au lecteur.

ON A GEOMETRICAL INTERPRETATION OF THE FUNCTION $at^2 + bu^2 + ctu$, WITH SOME APPLICATIONS.

By Thomas Weddle.

In the expression $at^2 + bu^2 + ctu$, let a, b, c be constants and t, u linear functions of the variable coordinates x, y, z; so that $t = 0$, and $u = 0$, are the equations to two planes intersecting in a straight line (tu). There is no difficulty in interpreting the given function when the roots of the quadratic, $at^2 + bu^2 + ctu = 0$, are real; for if the roots are real and unequal, the value which $at^2 + bu^2 + ctu$ has at any point will be proportional to the rectangle of the perpendiculars from that point on two fixed planes; and if the roots are real and equal, it will be proportional to the square of the perpendicular from the point on a certain fixed plane. These cases therefore present no difficulty, and I shall consequently confine myself to the consideration of the case in which the roots are unreal.

If we remove the origin of coordinates to a point in the line (tu), the new plane of xy being perpendicular to the line (tu), and those of yz and xz coinciding with t and u respectively, then, h and k being certain constants,

$$t = hx \text{ and } u = ky;$$

and putting $\alpha = ah^2$, &c.,

$$at^2 + bu^2 + ctu = \alpha x^2 + \beta y^2 + \gamma xy.$$

Now, by turning the coordinate planes yz and xz round the axis of z, we can reduce $\alpha x^2 + \beta y^2 + \gamma xy$ to the form $mx'^2 + ny'^2$ in an infinite number of ways; and since $at^2 + bu^2 + ctu$, that is $\alpha x^2 + \beta y^2 + \gamma xy$, cannot be resolved into real factors, m and n must have the same sign (which I shall suppose to be positive).

Now, of the infinite variety of ways in which $at^2 + bu^2 + ctu$ can be transformed into $mx'^2 + ny'^2$, there is one in which $m = n$; in this case

$$at^2 + bu^2 + ctu = m(x'^2 + y'^2);$$

but if p and q be the perpendiculars from any point on the planes of $y'z$ and $x'z$, and θ be the inclination of these planes, then $p = x' \sin\theta$, and $q = y' \sin\theta$, therefore

$$at^2 + bu^2 + ctu = \frac{m}{\sin^2\theta}(p^2 + q^2).$$

Hence

(A) *If the roots of the quadratic* $at^2 + bu^2 + ctu = 0$ *be unreal, two planes may be drawn through the straight line* (tu) *such that the value which the function* $at^2 + bu^2 + ctu$ *has at any point shall be proportional to the sum of the squares of the perpendiculars from that point on these two planes.*

Again, of the innumerable ways in which $at^2 + bu^2 + ctu$ can be transformed into $mx'^2 + ny'^2$, there is one in which the planes of $y'z$ and $x'z$ are perpendicular to each other. In this case let δ denote the distance of any point $(x'y'z)$ from the axis of z (that is, the straight line (tu)), this distance being measured parallel to either of the planes,

$$z = \pm y' \sqrt{\left(\frac{n-m}{m}\right)},$$

(m being supposed less than n), then it is easy to shew that

$$\delta^2 = x'^2 + y'^2 + \frac{n-m}{m} y'^2 = \frac{mx'^2 + ny'^2}{m} = \frac{at^2 + bu^2 + ctu}{m},$$

therefore $$at^2 + bu^2 + ctu = m.\delta^2.$$

Moreover it will be observed that the circular sections of the elliptic cylinder

$$mx'^2 + ny'^2 = \text{constant},$$

that is, $$at^2 + bu^2 + ctu = \text{constant},$$

are parallel to the same planes,

$$z = \pm y' \sqrt{\left(\frac{n-m}{m}\right)}; \text{ hence,}$$

(B) *If the roots of the quadratic* $at^2 + bu^2 + ctu = 0$ *be unreal, the value which the function* $at^2 + bu^2 + ctu$ *takes at any point is proportional to the square of the distance of that point from the line* (tu)*, this distance being measured parallel to a fixed plane. Moreover this fixed directive plane is parallel to a circular section of the elliptic cylinder* $at^2 + bu^2 + ctu$ = constant.*

Hence to find the directive plane, we have only to draw a plane to cut the elliptic cylinder $at^2 + bu^2 + ctu$ = constant in a circle; there are of course various ways of doing this, the following being one.

Perpendicular to the generators of the cylinder draw a plane which will cut the cylinder in an ellipse and the straight line (tu) in a point, the centre (O) of the ellipse. Draw the semimajor and semiminor axes OA and OB, and find one of the foci F. Along the axis of the cylinder (*i.e.* the straight line (tu)), and on each side of the plane of the ellipse, set off the distances Of and Of' each $= OF$; through B draw a straight line BC parallel to OA; and finally through BC and the points f and f' respectively draw planes; either of these may be taken for the directive plane. Also, if we find the equal conjugate diameters of the preceding ellipse and draw a plane through the straight line (tu), and each of these diameters, we shall have the two planes mentioned in (A).

It thus appears that if we can find the magnitude and position of the axes of the ellipse in which the cylinder $at^2 + bu^2 + ctu$ = constant is cut by a plane perpendicular to the straight line (tu), we can find both the two planes mentioned in (A) and the directive plane mentioned in (B).

It is clear that the converse of (A) is true; that is, if $t = 0$ and $u = 0$ be the equations to any two planes through the straight line (tu), then the sum of the squares of the perpendiculars from any point on any two fixed planes passing through this straight line is of the form $at^2 + bu^2 + ctu$, the roots of the quadratic $at^2 + bu^2 + ctu = 0$ being unreal; for

* Mr. Salmon informs me that he has already made use of this interpretation of the function $at^2 + bu^2 + ctu$ in connexion with the late Professor MacCullagh's method of generating surfaces of the second degree, but I have not seen Mr. Salmon's investigations.

these perpendiculars will be of the forms $\alpha t + \beta u$ and $\alpha' t + \beta' u$, and $(\alpha t + \beta u)^2 + (\alpha' t + \beta' u)^2$ is of the form $at^2 + bu^2 + ctu$.

Also, the square of the distance of any point from a fixed straight line, measured parallel to a fixed directive plane, is of the form $at^2 + bu^2 + ctu$, where $t = 0$ and $u = 0$ are the equations to any two planes passing through the straight line, and the roots of $at^2 + bu^2 + ctu = 0$ are unreal. For transform the coordinate axes so that the planes t and u may be taken for those of yz and xz, and the directive plane for that of xy; also let θ be the inclination of the axes of x and y, that is, of the straight lines in which the planes t and u intersect the directive plane. The square of the distance (δ) of any point (xyz) from the axis of z, measured parallel to the plane of xy, is

$$\delta^2 = x^2 + y^2 + 2xy \cos\theta.$$

Let us now restore the axes to their original position, then h and k being certain constants,

$$x = ht \text{ and } y = ku;$$

substituting these in the value of δ^2, we get a result of the form

$$\delta^2 = at^2 + bu^2 + ctu,$$

in which the roots of $at^2 + bu^2 + ctu = 0$ are evidently unreal.

When $c = 0$, so that δ^2 is of the form $at^2 + bu^2$, the planes t and u may be termed *conjugate* planes; and the preceding values of δ^2 inform us under what condition the planes t and u are conjugate, for if $c = 0$, then we must have $\cos\theta = 0$, or $\theta = 90°$, so that

If the two planes t and u be conjugate, they will cut the directive plane in straight lines at right angles to each other; and conversely, if two planes cut the directive plane in straight lines at right angles, they will be conjugate.*

Given the directive plane, we can therefore find the rectangular conjugate planes in the following manner. Draw a plane through the given straight line (tu) perpendicular to the directive plane, and through the same straight line draw another plane perpendicular to that just drawn, the two planes thus found are those required; for they are perpendicular to each other, and cut the directive plane in straight lines also perpendicular to each other.

* It is plain from what has been previously said that conjugate planes pass through conjugate diameters of the ellipse in which the cylinder $at^2 + bu^2 + ctu$ = constant is cut by a plane perpendicular to its generators, the *rectangular* conjugate planes passing through the axes.

Also, since the two planes mentioned in (A) are equally inclined to the planes just drawn, and cut the directive planes in straight lines at right angles to one another, we can construct them as follows:—Project the line (tu) orthogonally on the directive plane and let this projection be OP, O being the point in which the straight line (tu) intersects the directive plane; in this plane draw the straight lines OQ and OR on opposite sides of OP, each making half a right angle with it; and finally through the straight line (tu), and each of the lines OQ and OR, draw a plane. The two planes thus found are evidently those required.

Conversely, when the two planes mentioned in (A) are given, we can find the directive plane mentioned in (B). It is clear that the rectangular conjugate planes bisect the dihedral angles (one acute and the other obtuse) contained by the two given planes, and that the directive plane is perpendicular to that which bisects the obtuse angle; consequently the directive plane must cut the plane (Z) which bisects the acute angle in a straight line (OQ) at right angles to the line (OP) of intersection of the two given planes. Take O for the origin of rectangular axes, OP and OQ being the axes of z and x; also let 2θ denote the acute dihedral angle contained by the two given planes, then it is clear that the equations to these planes are

$$y = \tan\theta.x, \text{ and } y = -\tan\theta.x.$$

Also, since the directive plane passes through the axis of x, its equation will be

$$y = \tan\phi.z,$$

where ϕ is the inclination of the directive plane to the plane of xz. Hence the equations to the straight lines in which the directive plane intersects the given planes are

$$\frac{x}{\cot\theta} = y = \frac{z}{\cot\phi},$$

and

$$\frac{x}{-\cot\theta} = y = \frac{z}{\cot\phi};$$

but these two straight lines are at right angles (since the given planes are conjugate), therefore

$$-\cot^2\theta + 1 + \cot^2\phi = 0,$$

which reduces to

$$\sin\phi = \pm\tan\theta.$$

Hence we can find the directive plane by the following construction:—Draw a plane z bisecting the acute dihedral angle (2θ) of the given planes, in this plane draw a straight

line OQ perpendicular to the line of intersection of the said planes; and through OQ draw a plane on each side of z, the sine of whose inclination to z shall be equal to $\tan\theta$; then either of the planes thus drawn may be taken as the directive plane.

As a first application of these principles, let us take the equation to an ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(c+z)(c-z)}{c^2}.$$

Now, by (B), $\frac{x^2}{a^2}+\frac{y^2}{b^2}$ is proportional to the square of the distance of the point (xyz) from the axis of z, this distance being measured parallel to a circular section of the cylinder

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Hence

I. *If from any point in an ellipsoid a perpendicular be drawn to any of the axes, the rectangle of the segments of the axis is proportional to the square of the distance of the point from the axis, this distance being measured parallel to a circular section of that circumscribed cylinder whose generators are parallel to the axis.*

It is clear that this theorem will still be true, if instead of the axis we substitute any diameter of the ellipsoid, providing the segments of the diameter be made, not by a perpendicular but by a plane drawn through the point conjugate to the diameter. If the diameter be not an axis, the most interesting case is when it is conjugate to a circular section of the ellipsoid, for this being also one of the circular sections of the corresponding circumscribed cylinder, the preceding theorem will then take this form:

II. *If from any point in an ellipsoid a straight line be drawn parallel to a circular section so as to meet the diameter conjugate to that section, then shall the square of the straight line be proportional to the rectangle of the segments of the diameter.*

If we had applied (A) instead of (B) to interpret the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(c+z)(c-z)}{c^2},$$

we should have got the following theorem instead of (I).

If from any point in an ellipsoid a perpendicular be drawn to any axis, the rectangle of the segments of the axis is proportional to the sum of the squares of the perpendiculars from the point on two fixed planes passing through the axis.

The two fixed planes can, as we have seen, be constructed by means of the circumscribed cylinder. The interpretation (A), however, is in general so much less interesting than (B), that I shall take no further notice of the former, but shall leave the reader to modify each of the following theorems by using (A) instead of (B), it being remembered that the same cylinder or ellipse that determines the directive plane, also enables us to find the two planes mentioned in (A).

Again, the equation to the umbilical hyperboloid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(z+c)(z-c)}{c^2}.$$

III. *If from any point in an umbilical hyperboloid a straight line be drawn perpendicular to the real axis (produced), the rectangle of the segments of the axis is proportional to the square of the distance of the point from the axis, this distance being measured parallel to a certain fixed plane.*

This of course may be modified for any (real) diameter, as in the case of the ellipsoid. As z = constant, in the equation to the hyperboloid, implies $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ = constant, it follows that the cylinder of which the 'certain fixed plane' is a circular section will have for its base any section of the hyperboloid made by a plane perpendicular to the axis (or, more generally, conjugate to the diameter), its generators being parallel to the axis (or diameter, as the case may be). The following is evident.

IV. *If from any point in an umbilical hyperboloid a straight line be drawn parallel to a circular section, so as to meet the diameter (produced) conjugate to that section, then shall the square of the straight line be proportional to the rectangle of the segments of the diameter.*

Again, the equation to the elliptic paraboloid being

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{p},$$

we have the following theorem:

V. *If from any point in an elliptic paraboloid a straight line be drawn perpendicular to the axis, the portion of the axis intercepted between its vertex and the perpendicular is proportional to the square of the distance of the point from the axis, this distance being measured parallel to a certain fixed plane.*

This may be generalized for any diameter as in the ellipsoid and umbilical hyperboloid; and the cylinder which determines the fixed plane may be constructed precisely as in the case of the hyperboloid. When the diameter is conjugate to either series of circular sections, we have the following theorem:

VI. *If from any point in an elliptic paraboloid a straight line be drawn parallel to a circular section, so as to meet the diameter conjugate to that section, then shall the square of this straight line be proportional to the segment of the diameter intercepted between its vertex and the aforesaid straight line.*

It is scarcely necessary to remark that theorems (I), (III) and (V), (and consequently (II), (IV) and (VI),) are analogous to familiar and fundamental properties of the ellipse, hyperbola, and parabola, respectively.

The equation to a cone of the second degree is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2},$$

where the plane of xy is any plane through the vertex not intersecting the cone in any other real point, and the axis of z is the conjugate or reciprocal line. Hence

VII. *The distance of any point in a cone of the second degree from any plane passing through the vertex but not intersecting the cone in any other real point, has a constant ratio to its distance from the line reciprocal to the plane, this distance being measured parallel to a certain fixed plane.*

The cylinder which determines the last-mentioned plane has for base any section of the cone by a plane parallel to the plane through the vertex, and its generators are parallel to the reciprocal line.

When the plane through the vertex of the cone is parallel to a circular section, it is clear that the distance of the point from the line is to be measured parallel to that section.

Let there be a ruled surface of the second degree, and AB a straight line not intersecting the surface in real points. In this straight line take any point A, and draw its polar

plane intersecting AB in B: draw the polar plane of B which will pass through A and cut the former plane in a straight line CD; in this line take any point C and find its polar plane which will pass through AB and cut CD in some point D; then the tetrahedron $ABCD$ will be such that each angle is, with respect to the ruled surface, the pole of the opposite face. Hence if the faces BCD, ACD, ABD, and ABC, be denoted by $t = 0$, $u = 0$, $v = 0$, and $w = 0$, respectively, the equation to the surface of the second degree will be of the form

$$at^2 + bu^2 + cv^2 + ew^2 = 0;$$

but since the surface is *ruled*, two of the quantities a, b, c, e must be positive and two negative; also since the straight line AB or (vw) does not intersect the surface in real points, a and b must have the same sign; hence the preceding equation may be written

$$at^2 + bu^2 = fv^2 + gw^2,$$

where the constants a, b, f, and g are positive; or supposing these constants to be implicitly contained in t, u, v, and w, the equation may be written

$$t^2 + u^2 = v^2 + w^2.$$

Recollecting that the straight lines (tu) and (vw) are reciprocal, this equation gives us the following theorem:

VIII. *The distance of any point in a ruled surface of the second degree from a straight line which does not intersect the surface in real points is proportional to its distance from the reciprocal line, each distance being measured parallel to a fixed plane.*

We have now to determine the fixed planes (one corresponding to each line). Having constructed the planes t, u, v, and w, as before, we have

$$t^2 + u^2 = v^2 + w^2$$

for the equation to the surface; put $v = 0$, and we get

$$t^2 + u^2 = w^2,$$

the equation to the enveloping cone having its vertex at the angular point (tuw); next let w = constant = c, and we get

$$t^2 + u^2 = c^2,$$

which is the equation to the cylinder by means of which we have to find the fixed plane corresponding to the straight line (tu). Hence we may proceed as follows.

Find the sides in which the enveloping cone whose vertex is at (tuw), cuts the planes t and u, (or, in other words, join the point (tuw) to each of the points in which the straight lines (tv) and (uv) intersect the given surface); cut these sides by a plane parallel to w, and through the four points of section draw parallels to the straight line (tu); then a plane drawn perpendicular to (tu) will cut these parallels in points which are the extremities of two conjugate diameters of the ellipse in which the cylinder, $t^2 + u^2 = c^2$, is cut by a plane perpendicular to its generators. Having thus two conjugate diameters of this ellipse, the axes can be constructed and the directive plane found (as at p. 69). In a similar manner the directive plane corresponding to the straight line (vw) may be obtained.

If the surface of the second degree instead of being ruled be umbilical, then, having found the planes t, u, v, and w, as before, three of these planes will intersect the surface, and the fourth will not; let this last be the plane w, then the equation to the surface will be

$$t^2 + u^2 + v^2 = w^2,$$

therefore

$$t^2 + u^2 = (w + v)(w - v).$$

Now $w + v = 0$ and $w - v = 0$ are the equations to the two planes touching the surface at the extremities of the chord (tu); hence the following theorem, of which, by-the-by, (I), (III), and (V) may be considered particular cases:

IX. *The rectangle of the perpendiculars from any point in an umbilical surface of the second degree on two tangent planes is proportional to the square of the distance of the point from the chord of contact, this distance being measured parallel to a certain fixed plane.**

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* In an ingenious paper in this *Journal* Mr. Walker has given the following theorem (see vol. VII. p. 27):

"If two planes touch a surface of the second order in two points, the rectangle under perpendiculars let fall from any point on the surface on the planes is to the square of the perpendicular let fall on the chord of contact in a constant ratio."

By comparing this theorem with (IX) in the text, it is evident however that it is true only for those positions of the tangent planes in which the fixed (or directive) plane is perpendicular to the chord; (in this case the cylinder $t^2 + u^2 = c^2$ is of revolution). A more general theorem given by Mr. Walker is also liable to the same objection, and it therefore requires a little modification. One way of making this modification is as follows:

Let two surfaces of the second degree have double contact, and let $P = 0$ be the equation to one of them; then, by suitably drawing two planes $t = 0$ and $u = 0$ through the chord of contact, the equation to the

The fixed plane may be constructed precisely as in the case of theorem (VIII), taking care to make (*tuw*) and not (*tuv*) the vertex of the enveloping cone (for the cone $t^2 + u^2 + v^2 = 0$ is imaginary).

This theorem is perfectly analogous to a property of the conic sections, but it must be carefully noted that it applies to umbilical surfaces only, for the fixed or directive plane becomes imaginary in the case of the ruled surfaces.

It would not be difficult to give other applications of (B), but enough has been done for my present purpose, and I shall therefore conclude with one or two inferences from theorem (IX), which are analogous to certain plane properties.

Let $A_1A_2A_3 \ldots\ldots A_n$ be any polygon, plane or twisted, inscribed in an umbilical surface of the second degree; also let $p_1, p_2, p_3 \ldots \ldots p_n$ be the perpendiculars drawn from any point in the surface on the tangent planes at $A_1, A_2, A_3 \ldots A_n$; and $\delta_{12}, \delta_{23}, \delta_{34} \ldots \delta_{n1}$, the distances of the same point from the chords or sides A_1A_2, A_2A_3, $A_3A_4 \ldots A_nA_1$, these distances being measured parallel to certain planes which are to be constructed in the manner previously shewn. Then, $a, b, c, \ldots l$ being certain constants, we have, by (IX),

$$\left.\begin{aligned} p_1p_2 &= a.\delta_{12}^2, \\ p_2p_3 &= b.\delta_{23}^2, \\ p_3p_4 &= c.\delta_{34}^2, \\ &\vdots \qquad\quad \vdots \\ p_np_1 &= l.\delta_{n1}^2, \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (\alpha).$$

Multiplying these equations and extracting the square root, we have a result of the form

$$p_1p_2p_3 \cdots p_n = m \,.\, \delta_{12}.\delta_{23}.\delta_{34} \ldots \delta_{n1},$$

which gives this theorem:

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other surface may be exhibited under one of these forms,

$$P + t^2 + u^2 = 0 \quad \text{or} \quad P + t^2 - u^2 = 0.$$

Now in the former case the two surfaces intersect in imaginary conics in the imaginary planes $t + u\sqrt{(-1)} = 0$ and $t - u\sqrt{(-1)} = 0$; and in the latter they intersect in two conics, real or imaginary, in the real planes $t + u = 0$, and $t - u = 0$; hence, calling these planes (as M. Chasles has done) the *symptotic* planes of the two surfaces, we have the following theorem:

Let two surfaces of the second degree have double contact; when their symptotic planes are real, the rectangle of the segments of the chord (or secant) to one of these surfaces drawn through any point in the other parallel to a fixed line, is proportional to the rectangle of the perpendiculars from that point on the symptotic planes; but when the symptotic planes are imaginary, the first rectangle is proportional to the square of the distance of the point from the chord of contact, this distance being measured parallel to a certain fixed plane.

X. *If any polygon, either plane or twisted, be inscribed in an umbilical surface of the second degree, then planes may be found (one corresponding to each side) such that the continued product of the perpendiculars from any point in the surface on the tangent planes at the angles is proportional to the continued product of the distances of the said point from the sides, these distances being measured parallel to the planes previously found.*

Again, when the number of sides of the polygon is even, multiplying the first, third, fifth, &c. of equations (α), and also the second, fourth, sixth, &c., it is easy to see that we get a result of the form

$$\delta_{12}.\delta_{34}.\delta_{56}\ldots\ldots = m.\delta_{23}.\delta_{45}.\delta_{67}\ldots\ldots$$

Hence,

XI. *If an even-sided polygon, either plane or twisted, be inscribed in an umbilical surface of the second degree, planes may be found (one corresponding to each side) such that the continued product of the distances of any point in the surface from one set of alternate sides is proportional to the continued product of the distances of the same point from the other set of alternate sides, these distances being measured parallel to the planes previously found.*

In a plane the analogous theorem may be extended to an odd-sided polygon, provided we suppose the side wanting to be supplied by a tangent at one of the angles; and exactly in the same way (XI) may be extended to an odd-sided polygon, provided we substitute the perpendicular from the point in the surface on the tangent plane at one of the angles instead of the distance of the said point from the deficient side. Thus, suppose the polygon to have five sides we shall, from equations (α), get a result of the form

$$\delta_{12}.\delta_{34}.\delta_{51} = m.\delta_{23}.\delta_{45}.p_1,$$

so that $\delta_{12}.\delta_{34}.\delta_{51}$ is proportional to $\delta_{23}.\delta_{45}.p_1$.

It is worthy of observation that the plane theorem analogous to (XI) is convertible in the case of the quadrilateral, but that (XI) is not; for the locus of a point the product of whose distances from two opposite sides of a quadrilateral has a constant ratio to the product of its distances from the other two opposite sides is, in a plane, a conic circumscribed about the quadrilateral; while in three dimensions it is in general a surface of the fourth degree.

York Town, near Bagshot,
December 9, 1852.

ON A NEW AND SIMPLE RULE FOR APPROXIMATING TO THE AREA OF A FIGURE BY MEANS OF SEVEN EQUIDISTANT ORDINATES.

By Thomas Weddle.

Let $\beta, \beta_1, \beta_2, \ldots \beta_7$ be the values of the ordinate y when $x = 0, h, 2h, \ldots 6h$ respectively, or when $z = 0, 1, 2, \ldots 6$ respectively, where $z = \frac{x}{h}$. By a well-known formula we have

$$y = \beta + z.\Delta\beta + z(z-1).\frac{\Delta^2\beta}{2} + z(z-1)(z-2).\frac{\Delta^3\beta}{2.3} + \&c.$$

$$= \beta + z.\Delta\beta + (z^2 - z).\frac{\Delta^2\beta}{2} + (z^3 - 3z^2 + 2z).\frac{\Delta^3\beta}{2.3} + \&c.$$

Multiply by $\frac{dx}{h} = dz$, and integrate, therefore

$$\frac{1}{h}\int y dx = z\beta + \frac{z^2}{2}.\Delta\beta + \left(\frac{z^3}{3} - \frac{z^2}{2}\right)\frac{\Delta^2\beta}{2}$$

$$+ \left(\frac{z^4}{4} - \frac{3z^3}{3} + \frac{2z^2}{2}\right).\frac{\Delta^3\beta}{2.3} + \left(\frac{z^5}{5} - \frac{6z^4}{4} + \frac{11z^3}{3} - \frac{6z^2}{2}\right).\frac{\Delta^4\beta}{2.3.4}$$

$$+ \left(\frac{z^6}{6} - \frac{10z^5}{5} + \frac{35z^4}{4} - \frac{50z^3}{3} + \frac{24z^2}{2}\right).\frac{\Delta^5\beta}{2.3.4.5}$$

$$+ \left(\frac{z^7}{7} - \frac{15z^6}{6} + \frac{85z^5}{5} - \frac{225z^4}{4} + \frac{274z^3}{3} - \frac{120z^2}{2}\right).\frac{\Delta^6\beta}{2.3\ldots6}$$

$$+ \left(\frac{z^8}{8} - \frac{21z^7}{7} + \frac{175z^6}{6} - \frac{735z^5}{5} + \frac{1624z^4}{4} - \frac{1764z^3}{3} + \frac{720z^2}{2}\right).\frac{\Delta^7\beta}{2.3\ldots7}$$

$$+ \left(\frac{z^9}{9} - \frac{28z^8}{8} + \frac{322z^7}{7} - \frac{1960z^6}{6} + \frac{6769z^5}{5} - \frac{13132z^4}{4} + \frac{13068z^3}{3} - \frac{5040z^2}{2}\right).\frac{\Delta^8\beta}{2.3\ldots8}$$

$$+ \&c. \ldots\ldots (A).$$

Take this integral between the limits $z = 0$ and $z = 6$ (which correspond to the limits $x = 0$ and $x = 6h$), and multiply by h, therefore

$$\int_0^{6h} y\,dx = \left\{6\beta + 18.\Delta\beta + 27.\Delta^2\beta + 24.\Delta^3\beta + \frac{123}{10}.\Delta^4\beta + \frac{33}{10}.\Delta^5\beta + \frac{41}{140}\Delta^6\beta \ldots - \frac{9}{1400}\Delta^8\beta + \&c.\right\} h \ldots\ldots (B).$$

Now in finding the area of a figure by means of seven equidistant ordinates, we must suppose sixth differences constant, hence all the terms in (B) after $\Delta^6\beta$ will vanish; also since $\frac{41}{140}$ differs from $\frac{42}{140} = \frac{3}{10}$ by the small fraction $\frac{1}{140}$ only, and $\Delta^6\beta$ will usually be small, we may write $\frac{3}{10}\Delta^6\beta$ instead of $\frac{41}{140}\Delta^6\beta$, without material error; and then (B) takes the form

$$\int_0^{6h} y.dx = \{6\beta + 18.\Delta\beta + 27.\Delta^2\beta + 24.\Delta^3\beta + \frac{123}{10}.\Delta^4\beta + \frac{33}{10}.\Delta^5\beta + \frac{3}{10}\Delta^6\beta\}\, h \ldots\ldots(C).$$

For $\Delta\beta$, $\Delta^2\beta$, &c. write their values $\beta_1 - \beta$, $\beta_2 - 2\beta_1 + \beta$, &c., and (C) reduces to

$$\int_0^{6h} y.dx = \frac{3h}{10}\{\beta + \beta_2 + \beta_4 + \beta_6 + 5(\beta_1 + \beta_5) + 6\beta_3\}\ldots(D),$$

from which the value of the integral $\int_0^{6h} y.dx$, that is, of the area required, can be computed; but it will be better to throw the right-hand member into the form

$$\frac{3h}{10}\{\beta_3 + \beta + \beta_2 + \beta_4 + \beta_6 + 5(\beta_1 + \beta_3 + \beta_5)\},$$

which gives us the following very simple rule for approximating to the area of a figure by means of seven equidistant ordinates.

To five times the sum of the even ordinates, add the fourth (or middle) ordinate, and all the odd ordinates, multiply this sum by the common distance between the ordinates and three tenths of the product will be the area required.

It appears from the preceding investigation that this rule will give the *exact* area when *fifth* differences are constant (so that the sixth and subsequent differences vanish); while it differs (in excess) from the true value by only $\frac{1}{140}\Delta^6\beta$, when sixth (or even seventh) differences are constant. In other cases it will give the area very nearly *providing* the differences beginning at the sixth are small.

York Town, near Bagshot,
June 15, 1853.

ON THE TRANSFORMATION OF DIFFERENTIAL EQUATIONS.

By W. H. L. Russell, B.A.

In the differential equation

$$S\frac{dy}{dx} = P + Qy + Ry^2,$$

let P, Q, R be functions of (x) and contain moreover a certain quantity (n). If we put $y = \dfrac{u_0 + v_0 z}{u + vz}$ where u_0, v_0, u, v are functions of x and (n), the transformed equation will be of the form

$$S_0\frac{dz}{dx} = P_0 + Q_0 z + R_0 z^2,$$

P_0, Q_0, R_0, S_0 being functions of (x) and (n). Now we know that in certain specific cases S_0 after the transformation remains the same as S, while P_0, Q_0, R_0 are formed from P, Q, R, by merely increasing or diminishing (n) by unity. It becomes therefore interesting to ascertain what forms of the functions P, Q, R, S admit of this transformation. I propose in the following paper to shew how this may be done by certain examples. The transformations which I am about to give, besides being elegant in themselves, derive interest from their connexion with the theory of continued fractions, by which the functions which satisfy the differential equations which we are about to consider may always be expressed. An analogous method of investigation will also, I apprehend, apply to other classes of differential equations. I shall not in this paper write down the transformed differential equations, as well as those from which they were derived, but shall suppose the former always formed from the latter by changing (y) into (z), and (n) into $(n+1)$. Moreover, I shall always suppose the continued fractions I shall have occasion to refer to, to be of this form,

$$\frac{Ax^\mu}{1+}\quad\frac{Bx^\mu}{1+}\quad\frac{Cx^\mu}{1+}\ldots.$$

In the differential equation

$$s\frac{dy}{dx} = p + qy + ry^2,$$

let $y = \dfrac{x}{\alpha + \beta z}$, where α and β do not contain either (x) or

(z). Then the transformed equation will be

$$s\frac{dz}{dx} = \frac{s\alpha}{\beta x} - \frac{p\alpha^2}{\beta x} - \frac{q\alpha}{\beta} - \frac{rx}{\beta} + \left(\frac{s}{x} - \frac{2p\alpha}{x} - q\right)z - \frac{\beta p}{x}z^2.$$

Let
$$p = (A + Bx) + n(C + Ex),$$
$$q = (A' + B'x) + n(C' + E'x),$$
$$r = (A'' + B''x) + n(C'' + E''x),$$
$$s = L + Mx + Nx^2.$$

The transformed equation is to be formed from the original equation by changing (n) into $(n+1)$. Hence, we easily see that $L = 0,\quad A = 0,\quad C = 0,\quad B'' = 0,\quad E'' = 0.$

For the right-hand side of the transformed equation must contain no negative power of (x), and no positive power greater than unity. We shall, moreover, have the following equations obtained by equating like powers of (x), (we may put $M = 1$):

$$\frac{\alpha}{\beta} - \frac{\alpha^2}{\beta}(B + nE) - \frac{\alpha}{\beta}(A' + nC') = 0 \ldots\ldots\ldots\ldots (1),$$

$$\frac{\alpha}{\beta}N - \frac{\alpha}{\beta}(B' + nE') - \frac{1}{\beta}(A'' + nC'') = B + (n+1)E \ldots\ldots (2),$$

$$1 - 2\alpha(B + nE) - (A' + nC') = A' + (n+1)C' \ldots (3),$$

$$N - (B' + nE') = B' + (n+1)E' \ldots (4),$$

$$-\beta(B + nE) = A'' + (n+1)C'' \ldots (5).$$

From (1) we have
$$\alpha = \frac{1 - (A' + nC')}{B + nE}.$$

From (5) we have
$$\beta = -\frac{A'' + (n+1)C''}{B + nE}.$$

We must substitute these values in remaining equations, and equate coefficients of like powers of (n). Hence, we shall have

From (4), $\quad E' = 0,\quad N = 2B'.$

From (3), $\quad -1 + A' + nC' = A' + (n+1)C' \quad \text{or} \quad C' = -1.$

Also from (2) we shall have the following equation,

$$B'(1+n-A')-(B+nE)(A''+nC'')$$
$$=-\{B+(n+1)E\}\{A''+(n+1)C''\};$$
$$\therefore (1-A')B'+C''B+EA''+EC''=0, \quad B'+2EC''=0.$$

Hence, if $A''=0$, we have

$$C''=-\frac{B'}{2E}, \quad A'=\frac{E-B}{2E}.$$

Hence the equation may be written as follows:

$$x(1+2mx)\frac{dy}{dx}=(a+en)x+\left\{\frac{e-a}{2e}+mx-n\right\}y-\frac{mn}{2e}y^2,$$

$$y=\frac{x}{\alpha+\beta z}, \quad \text{where } \alpha=\frac{(2n+1)e+a}{2e(a+en)}, \quad \beta=\frac{(n+1)m}{2e(a+en)},$$

This equation can be completely integrated when (n) is an integer.

The following is obtained by assuming p, q, r to be of two dimensions in both x and (n), and proceeding as before,

$$x(1+mx^2)\frac{dy}{dx}=(a+bn+cn^2)x^2+\left(\frac{c-b}{c}+mx^2-2n\right)y-\frac{m}{c}y^2,$$

$$y=\frac{x^2}{\alpha+\beta z}, \quad \text{when } \alpha=\frac{(2n+1)c+b}{c(a+bn+cn^2)}, \quad \beta=\frac{m}{c(a+bn+cn^2)},$$

The known equations by which $\tan x$ and $\tan^{-1}x$ are expanded into continued fractions are particular cases of this. More generally,

$$x(1+mx^\mu)\frac{dy}{dx}=(a+bn+cn^2)x^\mu+\left(\frac{c-b}{2c}+\frac{mx^\mu}{2}-n\right)\mu y-\frac{\mu^2 m}{4c}y^2,$$

$$y=\frac{x^\mu}{\alpha+\beta z}, \quad \alpha=\frac{\mu(2cn+c+b)}{2c(a+bn+cn^2)}, \quad \beta=\frac{\mu^2 m}{4c(a+bn+cn^2)}.$$

This equation can be integrated whenever the equation $a+bx+cx^2=0$ has a positive entire root.

Let $\alpha=\phi(n)$, $\beta=\psi(n)$, then the continued fraction into which y can be expanded has only one period, and the coefficient of x^μ in it $=\dfrac{\psi(n)}{\phi(n)\,\phi(n+1)}$.

I shall now give an instance of a differential equation leading to a continued fraction with a double period, obtained by an analogous process:

$$x(1+mx)\frac{dy}{dx} = anx + \left(cx - 2n + \frac{ab+c}{m}\right)y + \left(b - \frac{mn}{a}\right)y^2,$$

$$y = \frac{\alpha x(1+\gamma z)}{1+\beta x+\gamma z},\quad \alpha = \frac{amn}{(2n+1)m-(ab+c)},\quad \gamma = \frac{m\{ab-m(n+1)\}}{a\{ab+c-2m(n+1)\}},$$

$$\beta = \frac{m\{(n+1)m-c\}\{(n+1)m-(ab+c)\}}{\{(ab+c)-(2n+1)m\}\{(ab+c)-2m(n+1)\}}.$$

This equation is always integrable when (n) is an integer. The known equation by which ε^x is expanded into a continued fraction is a particular case of it. Let

$$\alpha = \phi(n),\quad \beta = \psi(n),\quad \gamma = \theta(n).$$

Then the coefficients of x in the continued fraction by which (y) may be expressed are alternately of the forms $\psi(n)$ and $\theta(n)\,\phi(n+1)$.

The following is an example of a similar equation:

$$mx^2\frac{dy}{dx} = (k-an)x + (cx-2)y + by^2,\quad \text{when } m = \frac{ab}{2},$$

$$y = \frac{\alpha x(1+\gamma z)}{1+\beta x+\gamma z},\quad \alpha = \frac{k-an}{2},\quad \gamma = -\frac{b}{2},$$

$$\beta = \frac{(n+1)ab - (2c+bk)}{4}.$$

This equation can be completely integrated when the equation $k - an = 0$ has an entire positive root. The equation of Riccati may be made to depend on a particular case of this equation.

NOTE ON SIR JOHN WILSON'S THEOREM.

THE following is probably the best and the briefest mode of deducing Sir John Wilson's Theorem and its cognate Theorems from Fermat's. I can say nothing as to its originality.

p being any prime number, let

$$(x-1)(x-2)(x-3)\ldots\{x-(p-1)\} = x^{p-1} + A_1x^{p-2} + A_2x^{p-3} + \&c. + A_{p-1}.$$

Let x successively take the values $1, 2, 3\ldots(p-1)$; then to modulus p, by Fermat's theorem, we have

$$x^{p-1} + A_{p-1} \equiv 1 + A_{p-1},\ \text{say } A_0,$$

and we derive the $(p-1)$ congruences to modulus (p):

$$A_0 + A_1 + A_2 + A_3 \ldots\ldots\ldots\ldots + A_{p-2} \equiv 0,$$
$$A_0 + 2^{p-2}A_1 + 2^{p-3}A_2 + 2^{p-4}A_3 \ldots\ldots + 2A_{p-2} \equiv 0,$$
$$A_0 + 3^{p-2}A_1 + 3^{p-3}A_2 + 3^{p-4}A_3 \ldots\ldots + 3A_{p-2} \equiv 0,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$A_0 + (p-1)^{p-2}A_1 + (p-1)^{p-3}A_2 + (p-1)^{p-4}A_3 \ldots + (p-1)A_{p-2} \equiv 0.$$

Now the determinant formed by the coefficients of

$$A_0, A_1, A_2, \ldots\ldots A_{p-2}$$

is $1.2.3\ldots(p-1)$ multiplied into the product of the differences of $1, 2, 3, \ldots(p-1)$, and is therefore incongruent to zero for the modulus p. Hence, there being $(p-1)$ *independent* homogeneous congruences between $(p-1)$ quantities, each of these quantities must be congruent to zero, that is

$$A_0 \equiv 0, \quad A_1 \equiv 0, \ldots\ldots A_{p-2} \equiv 0 \ [\text{mod. } p].$$

The congruence $A_0 \equiv 0$, that is $1 + 1.2.3\ldots(p-1) \equiv 0 \ [\text{mod } p]$, is evidently Sir John Wilson's Theorem. We see also (by virtue of the remaining equations) at the same time, that the sums of the binary, ternary, &c., up to the $(p-2)^{\text{ary}}$ combinations of the numbers $1, 2, 3 \ldots (p-1)$, are all severally congruent to zero to the modulus p; *i.e.* are all divisible by that number.

J. J. S.

ON THE CALCULUS OF FORMS, OTHERWISE THE THEORY OF INVARIANTS.

By J. J. Sylvester.

Section VII. Continued.

Before proceeding further I must guard against a misconception as to my meaning to which the modification of the title of this memoir might give birth; it is not to be understood that I regard the Theory of Invariants as coextensive with the Calculus of Forms, but only with a certain portion of that Calculus which is here exclusively treated of; the Calculus of Forms itself has for its subject-matter the whole theory of the Composition, Decomposition, and Comparison of Forms. In the theory of invariants the composition of single forms with sets of linear forms is alone considered,

and the idea of invariance must be regarded as a transient idea arising out of an artificial mode of viewing the effects of composition, so as to ignore the presence in the result of factors which depend on the resultants of the linear forms employed, which resultants, although in this portion of the subject treated as a mere moduli and as such generally supposed to be reduced to unity, yet in regard to the general theory are as important as the factors which are retained as the sole objects of contemplation; so that in fact the idea of invariance is but a special and it may be said accidental notion which merges in the more general notion of permanency of character in the result of forms compounded in a given manner out of given forms. Again, as to combinants, the idea contained in this word may, by a change in the mode of statement of the definition, be extended to functions of unlike degrees. A combinant of $U, V, W, \ldots\ldots$ all functions of the same system or systems of variables is in fact only another name for invariants of the function $\lambda U + \mu V + \nu W +$ &c., where, over and above the sets of variables contained in $U, V, W, \ldots$ there is a new correlated set of variables λ, μ, ν, &c. So now, more generally, if $U, V, W, \ldots\ldots$ are of $p, q, r, \ldots\ldots$ dimensions in one set of variables of which the highest number is I if λ is taken of $I - p$, μ of $I - q$, W of $I - r$, &c. dimensions in the same, the functions λ, μ, ν, &c. being each the most general of their kind, any invariant of $\lambda U + \mu V + \nu W + \ldots\ldots$ which is such as well in respect to the coefficients in $\lambda, \mu, \nu, \ldots\ldots$ which must be considered as forming a set among themselves, as also in respect to the set of variables in $U, V, W, \ldots$ will be a combinant to the system $U, V, W, \ldots$; and so, more generally, if $U, V, W, \ldots\ldots$ contain several (say i) unrelated sets or systems of sets of variables, we must form in an analogous manner

$$\lambda_1\lambda_2\ldots\lambda_i U + \mu_1\mu_2\ldots\mu_i V + \nu_1\nu_2\ldots\nu_i W + \&c.,$$

and then an invariant in respect to the i given sets in $U, V, W, \ldots$ and the i new sets contained in $(\lambda_1, \mu_1, \nu_1, \ldots)$, $(\lambda_2, \mu_2, \ldots \nu_2)$, &c. $(\lambda_i, \mu_i, \nu_i, \ldots)$ will be a combinant to the system $U, V, W, \ldots$ Perhaps, however, a more immediate extension of the idea of combinants to the case supposed of i unrelated sets or systems of sets would be to take, instead of $\lambda_1\lambda_2\ldots\lambda_i$, $\mu_1\mu_2\ldots\mu_i$, &c., the perfectly general forms of the same degrees in each set of the variables as these quantities are respectively of the same; to use these general forms, the coefficients of which will constitute not i new sets but a

single new set of variables, as the syzygetic multipliers to $U, V, W, \ldots\ldots$ and then the invariant of the corresponding conjunctive in respect to the i original sets or systems of sets, and the one new set of variables thus obtained will be a combinant to the given system of Functions.* As a matter of punctilio I may here take the opportunity of observing that the process for obtaining the relation between $\mathfrak{W}$, $\mathfrak{w}$ (inadvertently written $\mathfrak{w}$), and R, would have been more perfectly symmetrical to the eye had the equation for W (p. 262) been written $\tau(z^2-y^2)=W$ in lieu of $\sigma(y^2-z^2)=W$. I now return to take up the subject from the point where it was brought to a close in the last number of the *Journal*.

Let us consider what the equation (A)† becomes when U, V, W becomes the first partial derivatives (quâ x, y, z) of a single homogeneous cubic function ψ, so that

$$U=\frac{d\psi}{dx}, \quad V=\frac{d\psi}{dy}, \quad W=\frac{d\psi}{dz},$$

$\mathfrak{H}$ then becomes the Hessian of ψ, and the S of this (like every other invariant of ψ)‡ may be expressed, as is well known, as a rational integral function of the S and T of ψ. The relation between the S of the H and the S and T may readily be obtained from the canonical form

$$(\psi)=x^3+y^3+z^3+6mxyz.$$

The Hessian of this is

$$(1+2m^3)\,xyz-m^2(x^3+y^3+z^3);$$

and making $-\dfrac{1+2m^3}{6m^2}=\mu$, the S of this Hessian will be

$$6^4.m^6\times(\mu-\mu^4),$$

* I propose to append at the end of the next or some subsequent Section what ought to have been given in this or previous place, viz. the general differential equations for any concomitant to any congeries of forms, comprising amongst them any number of various distinct (*i.e.* unrelated) classes of systems of sets of variables, the relations between the sets belonging to any one system being supposed to be either simple or compound, and after the manner of either cogredience or contragredience; in fact, to do this only requires a slight extension of the formulæ given by me with that object in the fifth section of my paper in the *Philosophical Transactions* for the year 1853, Part III., which see.

† *Vide* last number of this *Journal*, near the end of Author's paper therein.

‡ I have given a perfectly rigid demonstration in the *Philosophical Magazine*, in the early part of 1853, that every invariant to a cubic function of three variables is a rational integral function of the two Aronholdian invariants S and T.

which is $(1 + 2m^3)\{(1 + 2m^3)^3 + 216m^6\}$.

(See Calculus of Forms, *Camb. and Dub. Math. Journal*, Sect. 3, vol. VIII.)

that is
$$1 + 8m^3 + 240m^6 + 464m^9 + 16m^{12},$$
$$= (1 - 20m^3 - 8m^6)^2 + 48(m - m^4)^3,$$
$$= (S)^3 + 48(T)^3,$$

where (S) and (T) are respectively the S and T of (ψ). Hence we have in general

$$S.H.\psi = (S\psi)^3 + 48(T\psi)^3.$$

So that $\mathfrak{W}$ becomes $T^2 + 48S^3$, and $\mathfrak{V}$ evidently from Calculus of Forms (same page) becomes

$$\tfrac{1}{8}6(1 - 20m^3 - 8m^6),$$

that is $2T$, so that

$$4\mathfrak{W} - \tfrac{1}{4}\mathfrak{V}^2 = 3T^2 + 192S^3;$$

so that equation (A) becomes

$$R = T^2 + 64S^3,$$

the Aronholdian representation of the Discriminant of ψ.

We see from this numerical calculation that it is not $\Sigma\Omega$ but $\frac{1}{2}\Sigma\Omega$ which ought to receive the appellation of $\mathfrak{V}$, making which modification the general equation, written A, becomes

$$\tfrac{1}{3}R = 4\mathfrak{W} - \mathfrak{V}^2.$$

The $\mathfrak{W}$ it will be observed is a compound combinant, being a biquadratic function of quantities all of which are invariants of the system U, V, W; the $\mathfrak{V}$ on the other hand is a simple combinant of the sixth degree.

The general dodecadic combinant $\mathfrak{W}$ may also in another manner be exhibited as a biquadratic function of cubic functions of the coefficients of the three given quadratics; but these cubic functions will no longer be invariants of the given quadratics. Thus, form the Jacobian of U, V, W, that is, the determinant

$$\begin{vmatrix} \frac{dU}{dx}; & \frac{dU}{dy}; & \frac{dU}{dz} \\ \frac{dV}{dx}; & \frac{dV}{dy}; & \frac{dV}{dz} \\ \frac{dW}{dx}; & \frac{dW}{dy}; & \frac{dW}{dz} \end{vmatrix}$$

which will be a cubic covariant to the system. The S of this will be another form of $\mathfrak{w}$. So too, again, if we border the matrix to the Jacobian determinant above written vertically and horizontally with ξ, η, ζ, and call the determinant of the matrix thus formed I', I' will be quadratic in the system x, y, z, in the system ξ, η, ζ, and in the system formed by the coefficients of U, V, W, and the result of affecting this with the operator Σ will be the same as the result of the operation upon Ω with the same symbol; that is to say, $\frac{1}{24}E.I'$ will be equal to $\mathfrak{v}$, this latter symbol being so taken (as last explained) in such a way that $3R$ shall equal $4\mathfrak{w} - \mathfrak{v}^2$, and each of the four lines in the operator Σ being supposed to go their complete number (6) of permutations.

The terms sextic and dodecadic combinants will not be sufficient *per se* to characterize $\mathfrak{w}$ or $\mathfrak{v}$ (to a numerical factor *près*), supposing that there exist combinants of the 3rd and 9th degree respectively in the coefficients, in which case the general sextic would contain two and the general dodecadic five arbitrary numerical parameters.

This makes so much the more remarkable and satisfactory the method above developed for finding $\mathfrak{w}$ and $\mathfrak{v}$ as undecompounded forms; the general dodecadic combinant at all events being rendered indeterminate by virtue of the existence of a sextic combinant above demonstrated.

It is interesting to evince the identity of the S of the Jacobian with that of the discriminant to the conjunctive of U, V, W, which latter has been called $\mathfrak{w}$.

Starting with the canonical forms of the system U, V, W, and neglecting the ρ and σ, which cannot influence the result of the intended comparison, we have

$$J(U,V,W) = \begin{vmatrix} x; & -y; & 0 \\ 0; & -y; & z \\ gz+hy; & hx+y+fz; & gz+hy \end{vmatrix}$$

$$= fx(y^2+z^2) + gy(x^2+z^2) + hz(x^2+y^2) + xyz.$$

And multiplying by 6 and adopting the same notation as before (from the *Higher Plane Curves*, p. 182), we have

$$\begin{array}{lll} b_1 = 2f, & b_2 = 0, & b_3 = 2h, \\ a_1 = 0, & a_2 = 2g, & a_3 = 2h, \\ c_1 = 2f, & c_2 = 2y, & c_3 = 0, \\ & d = 1. & \end{array}$$

And the expression for S in *Higher Plane Curves*, p. 184, becomes, omitting every term containing a_1, b_2, or c_3,

$$d^4 - 2d^2(b_1c_1 + c_2a_2 + a_3b_3) + 3d(a_2b_3c_1 + a_3b_1c_2)$$
$$- (b_3c_2a_2a_3 + a_2c_1b_1b_3 + a_2b_1c_1c_2) + (b_1^2c_1^2 + c_2^2a_2^2 + a_3^2b_3^2),$$

that is $1 - 8(f^2 + g^2 + h^2) + 48fgh - 16(h^2y^2 + g^2f^2 + f^2h^2)$

$$+ 16(f^4 + g^4 + h^4), \quad \text{so that}$$

$$S_{x,y,z} J(U,V,W) = \mathfrak{w} = S_{\lambda,\mu,\nu} \square(\lambda U + \mu V + \nu W),$$

as was to be shewn. As observed above, the form first found has the advantage over the one just obtained in disclosing the elements (cubic invariants to U, V, W) of which the $\mathfrak{w}$ is a biquadratic function. So, analogously, the resultant of two quadratic functions (P, Q) of x and y may be exhibited either under the form of the discriminant in respect to the coefficients of conjunction of the discriminant in respect to the original variables of the conjunctive of P, Q, or under the form of the discriminant of the Jacobian of P, Q. The former discloses the invariantive composition of the resultant which remains latent in the latter. As regards the $\mathfrak{w}$, the proof of its being capable of the second mode of generation above indicated must, on account of the tediousness of the calculation, be for the present reserved; nor can I assert the fact with entire confidence until I have made a more complete investigation into the combinants of the system U, V, W, the remarks concerning which, in p. 262, I wish to be considered as provisionally withdrawn.

The analogy between the invariants of a cubic form of three variables and a biquadratic of two has been frequently insisted upon in the foregoing pages; but we shall now see that this analogy has its foundation in the deeper-seated analogy which connects a ternary system of quadratics of three variables with a binary system of cubics of two variables.

We may suppose the two given functions so combined that the linear conjunctive $lP + mQ$ shall contain two equal roots, and so take the form x^2y; this may then be combined with either of the given functions so as to give a conjunctive of the form

$$ax^3 + 3xy^2 + dy^3,$$

and writing for x and y, $\frac{x}{\sqrt{a}}, \frac{y}{\sqrt[3]{d}}$, respectively, and multiply-

ing $lP + mQ$ by $\sqrt[3]{a^2d}$, we obtain for our standard form

$$P = 3x^2y,$$

$$Q = x^3 + 3exy^2 + y^3.$$

The resultant of this system rejecting an universally-irrelevant numerical factor is 1.

Again, write

$$\lambda P + \mu Q = \lambda x^2y + \mu x^3 + 3\mu exy^2 + \mu y^3,$$

and operate upon this with the commutator (say ω),

$$\begin{vmatrix} \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \end{vmatrix}$$

Keeping one of the lines (*ex. gr.* the first) stationary, and, for greater brevity, writing $\delta_\lambda, \delta_\mu, \delta_x, \delta_y$ in place of $\frac{d}{d\lambda}, \frac{d}{d\mu}, \frac{d}{dx}, \frac{d}{dy}$, we obtain 8 positions, which, remembering that the order *in* the lines of these positions (and not the order *of* the lines) is the only thing to be attended to, are equivalent to

$$\begin{vmatrix} \delta_\lambda & \delta_\mu \\ \delta_x & \delta_y \\ \delta_x & \delta_y \\ \delta_x & \delta_y \end{vmatrix} - 3 \times \begin{vmatrix} \delta_\lambda & \delta_\mu \\ \delta_x & \delta_y \\ \delta_x & \delta_y \\ \delta_y & \delta_x \end{vmatrix} + 3 \times \begin{vmatrix} \delta_\lambda & \delta_\mu \\ \delta_x & \delta_y \\ \delta_y & \delta_x \\ \delta_y & \delta_x \end{vmatrix} - \begin{vmatrix} \delta_\lambda & \delta_\mu \\ \delta_y & \delta_x \\ \delta_y & \delta_x \\ \delta_y & \delta_x \end{vmatrix}$$

Hence we have $\frac{1}{3!}\omega(\lambda P + \mu Q) = -e$.

[I need hardly observe, that in general for any two odd-degreed functions of the same degree in x, y, as

$$a_0x^m + ma_1x^{m-1}.y + m.\tfrac{1}{2}(m-1)\, a_2x^{m-2}.y^2 + \ldots + m(a_1)\, xy^{m-1} + (a_0)\, y^m,$$

$$b_0x^m + mb_1x^{m-1}.y + m.\tfrac{1}{2}(m-1)\, b_2x^{m-2}.y^2 + \ldots + m(b_1)\, xy^{m-1} + (b_0)\, y^m,$$

we may obtain, in an analogous manner, the combinant

$$a_0(b_0) - ma_1(b_1) + m.\tfrac{1}{2}(m-1)\, a_2(b_2) + \&c.$$

Moreover it is easily shewn that when m is an even integer

the above expression will remain invariant, although of course it is no longer a combinant.]

Again, the Hessian to $\lambda P + \mu Q$ will be

$$\begin{vmatrix} \mu x + \lambda y, & \lambda x + \mu e y \\ \lambda x + \mu e y, & \mu e x + \mu y \end{vmatrix},$$

which is equal to

$$e\mu^2 x^2 + \mu^2 xy - e^2\mu^2 y^2 + \lambda\mu y^2 - e\lambda\mu xy - \lambda^2 x^2,$$

which call $H.C$ (C meaning the conjunctive of P, Q). Let this be operated upon with the commutator

$$\begin{matrix} \delta_x^2, & \delta_x\delta_y, & \delta_y^2, \\ \delta_\lambda^2, & \delta_\lambda\delta_\mu, & \delta_\mu^2, \end{matrix}$$

which call Ω.

Since neither $y^2\lambda^2$ nor $xy\lambda^2$ does not enter $H.C$, we have only to consider out of the full number 6 of positions the two effective positions

$$\begin{vmatrix} \delta_x^2 & \delta_x\delta_y & \delta_y^2 \\ \delta_\lambda^2 & \delta_\lambda\delta_\mu & \delta_\mu^2 \end{vmatrix} - \begin{vmatrix} \delta_x^2 & \delta_x\delta_y & \delta_y^2 \\ \delta_\lambda^2 & \delta_\mu^2 & \delta_\lambda\delta_\mu \end{vmatrix}$$

Hence $$\tfrac{1}{16} EHC(P,Q) = 1 - e^2.$$

So that $$\{-\tfrac{1}{36}\omega C(P,Q)\}^3 + \tfrac{1}{16} EHC(P,Q)\} = R(P,Q).$$

Thus R is expressed in terms of the cube of a simple quadratic combinant and a sextic compound combinant, which is made up of quadratic invariants. When P and Q become of the form $\frac{d\psi}{dx}$, $\frac{d\psi}{dy}$, respectively (ψ being a quartic in x and y), these become respectively (to numerical factors *près*) the quadrinvariant of the given function and the cube invariant of its Hessian, which latter is a linear function of the cube of the quadrinvariant and the square of the cubinvariant of the given function, as we know *à priori* from the fact of the fundamental scale of the quartic consisting of the quadrinvariant and cubinvariant (for a rigid demonstration of which fact see the *Philosophical Magazine* in the early part of 1853), and the expression for the resultant thus resolves itself into the known composite form of the sum of a square and cube.

The simple sextic combinant represented by $E.H.C(P,Q)$ may also, analogous to what has been observed concerning the $\mathfrak{B}$, be expressed as a commutant (in fact the cubinvariant)

of the Jacobian to P and Q, but then the form will no longer disclose its invariantive sub-composition. So too, if it were thought worth while to push the analogies to an extreme, the quadri-combinant to P, Q might have been found, first by bordering the Hessian to the conjunctive to P, Q with ξ, η horizontally and vertically, and operating upon the result with the commutator

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{d\lambda}, & \frac{d}{d\mu} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \end{vmatrix}$$

or by bordering the Jacobian to P, Q with ξ, η, as before, and then operating upon the result with the commutator

$$\begin{vmatrix} \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{dx}, & \frac{d}{dy} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \\ \frac{d}{d\xi}, & \frac{d}{d\eta} \end{vmatrix}$$

I propose hereafter to return to the consideration of the fundamental scale of combinants to the two systems, viz. of 3 quadratics in x, y, z, and of 2 cubics in x, y, which have been treated of in this section.

Section VIII.

On the Reduction of a Sextic Function of Two Variables to its Canonical Form.

In the *London and Edinburgh Philosophical Magazine* for Nov. 1851, after giving a simple method for representing any

function of two variables of an odd degree $(x, y)^{2m+1}$ under the form of

$$u_1^{2m+1} + u_2^{2m+} + \ldots\ldots + u_{m+1}^{2m+1},$$

where $u_1, u_2 \ldots u_{m+1}$ are linear functions of x, y (which form, as appears from the method of obtaining it, is unique). I proceeded to shew how by a certain method therein explained the biquadratic and octavic function of x, y, $(x, y)^4$, $(x, y)^6$ could be thrown under the respective forms

$$u_1^4 + u_2^4 + mu_1^2.u_2^2,$$

$$u_1^6 + u_2^6 + u_3^6 + u_4^6 + mu_1^2.u_2^2.u_3^2.u_4^2,$$

the number of values of m in the first form being 3 and in the second form 5, the quantity m in the one case depending on the solution of the equation

$$\begin{vmatrix} a_0 & a_1 & a_2 + \lambda \\ a_1 & a_2 - \frac{1}{2}\lambda & a_3 \\ a_2 + \lambda & a_3 & a_4 \end{vmatrix} = 0,$$

where a_0, a_1, a_2, a_3, a_4 are the coefficients of $(x, y)^4$ multiplied respectively by $1, \frac{1}{4}, \frac{1}{6}, \frac{1}{4}, 1$; and in the other case, on the solution of the equation

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 & a_4 + \lambda \\ a_1 & a_2 & a_3 & a_4 - \frac{1}{4}\lambda & a_5 \\ a_2 & a_3 & a_4 + \frac{1}{6}\lambda & a_5 & a_6 \\ a_3 & a_4 - \frac{1}{4}\lambda & a_5 & a_6 & a_7 \\ a_4 + \lambda & a_5 & a_6 & a_7 & a_8 \end{vmatrix} = 0,$$

$a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$,

being the coefficients of $(x, y)^8$ multiplied respectively by

$$1, \tfrac{1}{8}, \tfrac{1}{28}, \tfrac{1}{56}, \tfrac{1}{70}, \tfrac{1}{56}, \tfrac{1}{28}, \tfrac{1}{8}, 1.$$

Before proceeding to investigate the theory of these methods of reduction under any more general point of view, it will be convenient to seek to obtain the representation of $(x, y)^6$ under some analogous form.

It might at first be supposed that the corresponding form should be

$$u_1^6 + u_2^6 + u_3^6 + mu_1^2.u_2^2.u_3^2;$$

if, however, the method which succeeds for the quartic and octavic functions be attempted to be applied to this it will

be found entirely to fail. Here, however, considerations of a purely morphological character step in to our aid and immediately lead to the true canonical representation of the sextic function. Algebraically speaking, the only connexion between two identical forms F and F is through the equation $F = \psi^{-1}\psi F$; but, morphologically considered, a *form* F may admit of being derived by a series of entirely heterogeneous operations from *itself*. In general, supposing

$$F(x, y) = ax^n + nbx^{n-1}.y + \&c. \ldots + n(b)\, xy^{n-1} + (a)\, y^n,$$

the form $\xi^n \dfrac{d}{da} + \xi^{n-1}\eta \dfrac{d}{db} + \ldots + \xi\eta^{n-1} \dfrac{d}{d(b)} + \eta^n \dfrac{d}{d(a)}$,

operating upon any concomitant to F will, we know, (from the law of reciprocity in Section IV.) produce another concomitant. The operative form above written is termed the evector, and the result of operating therewith upon a concomitant is termed the evectant of the latter, which is said, when so operated upon, to be evected.* The polar reciprocal of the evector may be termed the contravector, and for two variables is of course of the form

$$y^n \frac{d}{da} - y^{n-1}.x \frac{d}{db} \pm \&c.$$

If we suppose n to be even, $F(x, y)$ will have the well-known quadrinvariant

$$a(a) - nb.(b) + n\tfrac{1}{2}(n-1)\, c.(c) \mp \&c.,$$

and if this be operated upon with the contravector, or if we like so to say, be contravected, we recover the original function F, so that any function of two variables of an even degree is the contravect of its quadrinvariant.

If now we return to the representation of $(x, y)^4$ under the form

$$u_1^4 + u_2^4 + m(u_1u_2)^2,$$

and make $u_1u_2 = F_2(x, y)$;

or to that of $(x, y)^5$ under the form

$$u_1^4 + u_2^4 + u_3^4 + u_4^4 + m(u_1u_2u_3u_4)^2,$$

* These terms "evector, evectant, contravectant, to evect and contravect," will of course admit of an immediate extension to functions of any number of variables. Evection gives rise to contravariants, contravection to covariants; but on this account to interchange the meanings respectively attached to the terms evector and contravector, and their respective allied terms, would be a simplification too dearly purchased at the expense of contravening the principle that the *word for the base should be the base for the word.*

and make $$u_1.u_2.u_3.u_4 = F_4(x, y),$$

the outstanding term multiplied by the parameter m may be regarded in each of these two cases as the squared contravects of the quadrintrivariants F_2 and F_4 respectively. Under this point of view we at once see a ground for the proved fact of $(x, y)^6$ not being capable of being thrown under the form

$$u_1^6 + u_2^6 + u_3^6 + m\{F_3(x, y)\}^2,$$

where $$u_1.u_2.u_3 = F_3(x, y),$$

because there exists no quadrinvariant to $F_3(x, y)$, the only invariant which it possesses being the discriminant which is of the fourth degree; if however instead of $m\{F_3(x, y)\}^2$ we write $mF_3(x, y)G_3(x, y)$, where $G_3(x, y)$ is the contravect of the discriminant of F_3, we shall find that the method applied to the reduction of $(x, y)^4$ and to $(x, y)^8$ will perfectly well succeed for $(x, y)^6$, as I proceed to demonstrate.

Let this function be written under the form

$$a_0x^6 + 6a_1x^5y + 15a_2x^4y^2 + 20a_3x^3y^3 + 15a_4x^2y^2 + 6a_5x^5y^4 + a_6y^6,$$

which suppose made equal to

$$(p_1x + q_1y)^6 + (p_2x + q_2y)^6 + (p_3x + q_3y)^6 + (Ax^3+2Bx^2y+3Cxy^2+Dy^3)\,(Lx^3+Mx^2y+Nxy^2+Py^3);$$

where $$(p_1x+q_1y)\,(p_2x+q_2y)\,(p_3x+q_3y) = Ax^3+3Bx^2y+3Cxy^2+Dy^3,$$

the discriminant of this will be, as is well known,

$$A^2D^2 + 4AC^3 + 4DB^3 - 3A^2D^2 - 6ABCD,$$

and contravecting this with the operator

$$\left(y^3\frac{d}{dA} - 3y^2x\frac{d}{dB} + 3yx^2\frac{d}{dC} - x^3\frac{d}{dD}\right);$$

and, identifying the result with $Lx^3 + 3Mx^2y + 3Nxy^2 + Py^3$, we have

$$L = 6ABC + 6A^2D - 4B^3,$$
$$M = -6ABD + 12AC^2,$$
$$N = 6ACD - 12DB^2,$$
$$P = -6BCD - 6AD^2 + 4C^3.$$

A, B, C, D are known functions of p_1, p_2, p_3; q_1, q_2, q_3, and we shall have 7 equations for determining these 6 unknown quantities and the unknown parameter m.

*Let $q_1 = p_1\lambda_1, \quad q_2 = p_2\lambda_2^4, \quad q_3 = p_3\lambda_3,$

$$\lambda_1 + \lambda_2 + \lambda_3 = 3s_1, \quad \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2 = 3s_2, \quad \lambda_1\lambda_2\lambda_3 = s_3,$$

$$p_1p_2p_3 = m.$$

Then $A = m, \quad 3B = 3ms_1, \quad 3C = 3ms_2, \quad D = s_3.$

$$L = m^3(4s_1^3 - 6s_1s_2 + 2s_3),$$

$$M = m^3(6s_1^2s_2 + 6s_1s_3 - 12s_2^2),$$

$$N = m^3(12s_1^2s_3 - 6s_1^2s_2 - 6s_2s_3),$$

$$P = m^3(+ 6s_1s_2s_3 - 4s_2^3 - 2s_3^2).$$

Let $(Ax^3 + 3Bx^2y + 3Cxy^2 + Dy^3)(Lx^3 + Mx^2y + Nxy^2 + Py^3)$

$= K_0x^6 + K_1x^5y + K_2x^4y^2 + K_3x^3y^3 + K_4x^2y^4 + K_5xy^5 + K_6y^6$

$= T.$

Then, equating this term for term with

$$\lambda_1^6(x + p_1y)^6 + \lambda_2^6(x + p_2y)^6 + \lambda_3^6(x + p_3y)^6 + \mu T,$$

we obtain the 7 equations following:

$$p_1^6 + p_2^6 + p_3^6 + \mu K_0 = a_0 \ldots\ldots\ldots (1),$$

$$p_1^6\lambda_1 + p_2^6\lambda_2 + p_3^6\lambda_3 + \frac{\mu}{6} K_1 = a_1 \ldots\ldots\ldots (2),$$

$$p_1^6\lambda_1^2 + p_2^6\lambda_2^2 + p_3^6\lambda_3^2 + \frac{\mu}{15} K_2 = a_2 \ldots\ldots\ldots (3),$$

$$p_1^6\lambda_1^3 + p_2^6\lambda_2^3 + p_3^6\lambda_3^3 + \frac{\mu}{20} K_3 = a_3 \ldots\ldots\ldots (4),$$

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* The reader will please to make the following correction in the text preceding this:—

All that comes between the word 'operator' (10th line from foot of p. 96) and the letters 'A, B, C, D' (3rd line from foot of same page) is to be omitted and the following inserted in its stead:

$$-y^3\frac{d}{dA} + y^2x\frac{d}{dB} - yx^2\frac{d}{dC} + x^3\frac{d}{dD},$$

and identifying the result with $Lx^3 + Mx^2y + Nxy^2 + Py^3$, we have

$$L = -6ABC + 2A^2D + 4B^3,$$

$$M = 6ABD - 12AC^2 + 6B^2C,$$

$$N = -6ACD + 12DB^2 - 6BC^2,$$

$$P = 6BCD - 2AD^2 - 4C^3.$$

$$p_1^6\lambda_1^4 + p_2^6\lambda_2^4 + p_3^6\lambda_3^4 + \frac{\mu}{15}K_4 = a_4 \ldots\ldots\ldots (5),$$

$$p_1^6\lambda_1^5 + p_2^6\lambda_2^5 + p_3^6\lambda_3^5 + \frac{\mu}{6}K_5 = a_5 \ldots\ldots\ldots (6),$$

$$p_1^6\lambda_1^6 + p_2^6\lambda_2^6 + p_3^6\lambda_3^6 + \mu K_6 = a_6 \ldots\ldots\ldots (7).$$

Eliminating linearly

p_1^6, p_2^6, p_3^6 between equations 1, 2, 3, 4,

$\lambda_1 p_1^6$, $\lambda_2 p_2^6$, $\lambda_3 p_3^6$ 2, 3, 4, 5,

$\lambda_1^2 p_1^6$, $\lambda_2^2 p_2^6$, $\lambda_3^2 p_3^6$ 3, 4, 5, 6,

$\lambda_1^3 p_1^6$, $\lambda_2^3 p_2^6$, $\lambda_3^3 p_3^3$ 4, 5, 6, 7,

we obtain the 4 equations following, viz.

$$a_0 s_3 - 3a_1 s_2 + 3a_2 s_1 - a_3 = \mu\vartheta_0,$$
$$a_1 s_3 - 3a_2 s_2 + 3a_3 s_1 - a_4 = \mu\vartheta_1,$$
$$a_2 s_3 - 3a_3 s_2 + 3a_4 s_1 - a_5 = \mu\vartheta_2,$$
$$a_3 s_3 - 3a_4 s_2 + 3a_5 s_1 - a_6 = \mu\vartheta_3,$$

where $\vartheta_0 = K_0 s_3 - \frac{3}{6}K_1 s_2 + \frac{3}{15}K_2 s_1 - \frac{1}{20}K_3$
$= \frac{1}{60}(60K_0 s_3 - 30K_1 s_2 + 12K_2 s_1 - 3K_3),$

$\vartheta_1 = \frac{1}{6}K_1 s_3 - \frac{3}{15}K_2 s_2 + \frac{3}{20}K_3 s_1 - \frac{1}{15}K_4$
$= \frac{1}{60}(10K_1 s_3 - 12K_2 s_2 + 9K_3 s_1 - 4K_4),$

$\vartheta_2 = \frac{1}{15}K_2 s_3 - \frac{3}{20}K_3 s_2 + \frac{3}{15}K_4 s_1 - \frac{1}{6}K_5$
$= \frac{1}{60}(4K_2 s_3 - 9K_3 s_2 + 12K_4 s_1 - 10K_5),$

$\vartheta_3 = \frac{1}{20}K_3 s_3 - \frac{3}{15}K_4 s_2 + \frac{3}{6}K_5 s_1 - K_6$
$= \frac{1}{60}(3K_3 s_3 - 12K_4 s_2 + 30K_5 s_1 - 60K_6),$

and

$$\frac{1}{m^4}K_0 = \frac{A}{m}\cdot\frac{L}{m^3} = 4s_1^3 - 6s_1 s_2 + 2s_3,$$

$$\frac{1}{m^4}K_1 = \frac{AM + 3BL}{m^4} = (6s_1^2 s_2 + 6s_1 s_3 - 12s_2^2) + 3s_1(4s_1^3 - 6s_1 s_2 + 2s_3)$$
$$= 12s_1^4 - 12s_1^2 s_2 + 12s_1 s_3 - 12s_2^2,$$

$$\frac{1}{m^4}K_2 = \frac{AN + 3BM + 3CL}{m^4} = (12s_1^2 s_3 - 6s_1 s_2^2 - 6s_2 s_3)$$
$$+ (18s_1^3 s_2 + 18s_1^2 s_3 - 36s_1 s_2^2)$$
$$+ (12s_1^3 s_2 - 18s_1 s_2^2 + 6s_2 s_3)$$
$$= 30s_1^3 s_2 + 30s_1^2 s_3 - 60s_1 s_2^2,$$

$$\frac{1}{m^4}K_3 = \frac{AP+3BN+3CM+DL}{m^4} = (6s_1s_2s_3 - 4s_2^3 - 2s_3^2)$$
$$+ (36s_1^3s_3 - 18s_1^2s_2^2 - 18s_1s_2s_3)$$
$$+ (18s_1^2s_2^2 + 18s_1s_2s_3 - 36s_2^3)$$
$$+ (4s_1^3s_3 - 6s_1s_2s_3 + 2s_3^2)$$
$$= 40s_1^3s_3 - 40s_2^3,$$

$$\frac{1}{m^4}K_4 = \frac{3BP+3CN+DM}{m^4} = 18s_1^2s_2s_3 - 12s_1s_2^3 - 6s_1s_3^2$$
$$+ 36s_1^2s_2s_3 - 18s_1s_2^3 - 18s_2^2s_3$$
$$+ 6s_1^2s_2s_3 + 6s_1s_3^2 - 12s_2^2s_3$$
$$= 60s_1^2s_2s_3 - 30s_1s_2^3 - 30s_2^2s_3,$$

$$\frac{1}{m^4}K_5 = \frac{3CP+DN}{m^4} = 18s_1s_2^2s_3^2 - 12s_2^4 - 6s_2s_3^2$$
$$+ 12s_1^2s_3^2 - 6s_1s_2^2s_3 - 6s_2s_3^2$$
$$= 12s_1^2s_3^2 + 12s_1s_2^2s_3 - 12s_2^4 - 12s_2s_3^2,$$

$$\frac{1}{m^4}K_6 = \frac{DP}{m^4} = 6s_1s_2s_3^2 - 4s_2^3s_3 - 2s_3^3;$$

$$\therefore \frac{60\vartheta_0}{\mu m^4} = 240s_1^3s_3 - 360s_1s_2s_3 + 120s_3^2$$
$$- 360s_1^4s_2 + 360s_1^2s_2^2 - 360s_1s_2s_3 + 360s_2^3$$
$$+ 360s_1^4s_2 + 360s_1^3s_3 - 720s_1^2s_2^2$$
$$- 120s_1^3s_3 + 120s_2^3$$
$$= 120(s_3^2 + 4s_2^3 + 4s_1^3s_3 - 3s_1^2s_2^2 - 6s_1s_2s_3),$$

i.e. $\vartheta_0 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD).$

Again,

$$\frac{\vartheta_1}{\mu m^4} = 120s_1^4s_3 - 120s_1^2s_2s_3 + 120s_1s_3^2 - 120s_2^2s_3$$
$$- 360s_1^3s_2^2 - 360s_1^2s_2s_3 + 720s_1s_2^3$$
$$360s_1^4s_3 - 360s_1s_2^2$$
$$- 240s_1^2s_2s_3 + 120s_1s_2^3 + 120s_1s_2^2s_3$$
$$= 120(s_1s_3^2 + 4s_1s_2^3 + 4s_1^4s_3 - 3s_1^3s_2^2 - 6s_1^2s_2s_3);$$

$$\therefore \vartheta_1 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)\,s_1.$$

Again,

$$\begin{aligned}\frac{60\vartheta_2}{\mu m^4} &= 120s_1^3s_2s_3 + 120s_1^2s_2^2 - 240s_1s_2^2s_3\\ &\quad - 360s_1^3s_2s_3 + 360s_2^4\\ &\quad + 720s_1^3s_2s_3 - 360s_1^2s_2^2 - 360s_1s_2^2s_3\\ &\quad - 120s_1^2s_3^2 - 120s_1s_2^2s_3 + 120s_2^4 + 120s_2s_3^2\\ &= 120(s_2s_3^2 + 4s_3^4 + 4s_1^3s_4s_3 - 3s_1^2s_2^2 - 6s_1s_2^3s_3),\end{aligned}$$

$$\therefore\quad \vartheta_2 = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)\,s_3.$$

Finally,

$$\begin{aligned}\frac{60\vartheta_3}{\mu m^4} &= 120s_1^3s_3^2 - 120s_2^3s_3\\ &\quad - 720s_1^2s_2^2s_3 + 360s_1s_2^4 + 360s_2^3s_3\\ &\quad + 360s_1^3s_3^2 + 360s_1^2s_2^2s_3 - 360s_1s_2^4 - 360s_1s_2s_3^2\\ &\quad - 360s_1^2s_3s_3^2 + 240s_2^3s_3 + 120s_3^3\\ &= 120(s_3^3 + 4s_2^3s_3 + 4s_1^3s_3^2 - 3s_1^2s_2^2s_3 - 6s_1s_2s_3^2),\end{aligned}$$

$$\therefore\quad \vartheta = 2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD)\,s_3.$$

Hence, writing

$$2\mu(A^2D^2 + 4AC^3 + 4DB^3 - 3B^2C^2 - 6ABCD) = \rho,$$

the four equations connecting a_0, a_1, a_2, a_3 with $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3$, take the form

$$\begin{aligned}a_0s_3 - 3a_1s_2 + 3a_2s_1 - (a_3 + \rho) &= 0,\\ a_1s_3 - 3a_2s_2 + 3(a_3 - \tfrac{1}{3}\rho)\,s_1 - a_4 &= 0,\\ a_2s_3 - 3(a_3 + \tfrac{1}{3}\rho)\,s_2 + 3a_4s_1 - a_5 &= 0,\\ (a_3 - \rho) - 3a_4s_2 + 3a_5s_1 - a_6 &= 0.\end{aligned}$$

Hence we derive the equations involving only the known coefficients of the given function for finding ρ, viz. the determinant

$$\begin{vmatrix} a_0 & a_1 & a_2 & a_3 + \rho\\ a_1 & a_2 & a_3 - \frac{1}{3}\rho & a_4\\ a_2 & a_3 + \frac{1}{3}\rho & a_4 & a_5\\ a_3 - \rho & a_4 & a_5 & a_6 \end{vmatrix} = 0 \ldots\ldots (\mathrm{R}).$$

If in this matrix ρ be changed into $-\rho$, the determinant evidently remains unaltered in value; hence the odd powers of ρ disappear from the equation, and ρ may be found by the solution of a double quadratic only. In fact the above

equation for finding ρ, expanded out, becomes

$$\frac{\rho^4}{9} + \left(\frac{2}{3}\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} - \frac{2}{3}\begin{vmatrix} a_1 & a_3 \\ a_3 & a_5 \end{vmatrix} + \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} + \frac{1}{9}\begin{vmatrix} a_0 & a_3 \\ a_3 & a_6 \end{vmatrix}\right)\rho^2,$$

$$- \begin{vmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_4 & a_5 \\ a_3 & a_4 & a_5 & a_6 \end{vmatrix} = 0;$$

that is

$$\rho^4 + (15a_2a_4 - 6a_1a_5 - 10a_3^2 + a_0a_6)\,\rho^2$$

$$+ \begin{vmatrix} a_3 & a_2 & a_1 & a \\ a_4 & a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 & a_2 \\ a_6 & a_5 & a_4 & a_3 \end{vmatrix} = 0;$$

the coefficient of ρ^2 being the well-known quadrinvariant, and the final term the meiocatalecticizant of the given function. There will consequently be four different values of ρ and four different systems of values of s_1, s_2, s_3, expressible for each system respectively in terms of ρ by means of any three out of the four equations (R), and consequently there will be four systems of values of $\lambda_1, \lambda_2, \lambda_3$, each of which may be found separately by solving the cubic equation

$$\lambda^3 - 3s_1\lambda^2 + 3s_2\lambda - s_3 = 0;$$

also $K_0, K_1, K_2, K_3, K_4, K_5, K_6$ become known multiples of m^4, and finally, the values of any λ and K system being thus determined, we may then, by means of the identity

$$p_1^6(x + \lambda_1 y)^6 + p_2^6(x + \lambda_2 y)^6 + p_3^6(x + \lambda_3 y)^6$$

$$+ \mu m^4\left(\frac{K_0}{m^4}x^6 + \frac{K_1}{m^4}x^5y + \&c. + \frac{K_6}{m^4}y^6\right) = a_0x^6 + 6a_1x^5y + \&c. + a_6y^6,$$

write down at will any 4 equations out of the 7 equations therefrom resulting, and these will serve to determine linearly the values of $p_1^6, p_2^6, p_3^6, \mu m^4$; and consequently, by means of the equations

$$q_1 = p_1\rho_1, \quad q_2 = p_2\rho_2, \quad q_3 = p_3\rho_3,$$

q_1, q_2, q_3 are known, and consequently every coefficient in

$$(p_1x + q_1y)^6 + (p_2x + q_2y)^6 + (p_3x + q_3y)^6 + \mu M$$

is completely determined. But we shall hereafter return to this theory, and seek for a direct method of finding the four values of the functions $(p_1x + q_1y), (p_2x + q_2y), (p_3x + q_3y)$.

It appears from the above investigation that there are four modes of throwing $(x, y)^6$ under the assumed form which possess the remarkable property of separating into two pairs of modes, as is obvious from the fact of the resolving equation in ρ having two pairs of roots, those of the same pair being equal but of contrary signs. As this form will be of extreme value in studying the invariants of $(x, y)^6$, it may be well to consider the simplest shape to which it admits of being reduced.

We may suppose $(p_1x + q_1y)(p_2x + q_2y)(p_3x + q_3y)$ thrown under the form of $u^3 + v^3$, the contravectant of the discriminant to which in respect to u and v is $v^3 - u^3$, so that we may use for the canonical form the expression

$$a(u + v)^6 + b(u + \rho v)^6 + c(u + \rho^2 v)^6 + \mu(u^6 - v^6),$$

where $\rho^3 = 1$; or if we please, more simply

$$a(u + v)^6 + b(u + \rho v)^6 + c(u + \rho^2 v)^6 + u^6 - v^6.$$

I may take this occasion to observe that there are generally two modes of a distinct kind for obtaining any simple concomitant ; the difference (a most important practical one) consisting in the circumstance that in the one mode there are differentiations to be performed in respect to the coefficients, the consequence of which is that the whole of the operations must be gone through for obtaining the concomitant the primitive in its most general form, and no advantage can be taken in the course of these operations of the simplification resulting from the absence of any terms in the primitive or of any other specialty therein ; whereas in the other mode of derivation, where all the differentiations have to be performed quâ the variables only, the partial form may be operated with throughout. Thus, for instance, to find the contravectant to the discriminant of a cubic function the general form of the cubic must be employed, and then the special values of the coefficient corresponding to a specific form of the cubic substituted at the close of the operations ; but this same concomitant may also be obtained by taking the resultant of the first emanant of the given cubic and of the first emanant of its Hessian in respect to the variables of emanation, and consequently the specific form may, after this mode, be retained from the first. Thus, if we start with $u^3 + v^3$, the Hessian is uv, and the two emanants in question will be $u^2u' + v^2v'$ and $vu' + uv'$, the resultant of which in respect to u' and v' is $u^3 - v^3$; or, again, if we commence with uvw subject to the relation that $u + v + w = 0$, the Hessian

will be

$$\begin{vmatrix} 0 & w & v & 1 \\ w & 0 & u & 1 \\ v & u & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

that is to say, $u^2 + v^2 + w^2 - 2uv - 2uw - 2vw$.

The two emanants will then be

$$vwu' + wuv' + uvw'$$

$$(u - v - w)u' + (v - w - u)v' + (w - u - v)w',$$

subject to the relation

$$u' + v' + w' = 0;$$

and taking the resultant of these three equations, or, which is the same thing, of

$$\begin{aligned} vwu' + wuv' + uvw', \\ uu' + vv' + ww', \\ u' + v' + w', \end{aligned}$$

we obtain the determinant

$$\begin{vmatrix} vw & wu & uv \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix}$$

which is equal to

$$vw(v - w) + wu(w - u) + uv(u - v),$$

that is to say $(u - v)(v - w)(w - u)$.

Hence another variety of the external shape to which the canonical form for the sextic function of x, y may be reduced will be

$$au^6 + bv^6 + cw^6 + \mu uvw(u - v)(v - w)(w - u).$$

I shall presently revert to the theory of the corresponding mode of reducing to their canonical forms the biquadratic and octavic functions of x, y, the number of solutions for which will be respectively 3 and 5, and the discovery of which, as shewn by me in the Number of the *Phil. Mag.* before adverted to, depend upon the solution of equations of the third and fifth degrees in ρ expressed by means of determinants of the third and fifth orders formed in precise correspondence with that of the fourth order, upon which, as we have found above, the reduction of the sextic function to its canonical form depends.

[*To be Continued.*]

ON THE INTEGRATION OF LINEAR DIFFERENTIAL EQUATIONS.

By W. H. L. Russell.

I PROPOSE in the following paper to investigate a method for solving an extensive class of differential equations by means of definite integrals. I shall commence with some theorems relative to the summation of series, of which I shall make continual use in the analysis on which I am about to enter.

To determine the sum of the series

$$1+\frac{\alpha}{\beta}x+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\cdot\frac{x^2}{1.2}+\frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)}\cdot\frac{x^3}{1.2.3}+\&c.,$$

$$\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\ldots(\beta+n-1)}=\frac{\Gamma\beta}{\Gamma\alpha\,\Gamma(\beta-\alpha)}\int_0^1 v^{\alpha+n-1}(1-v)^{\beta-\alpha-1}\,dv.$$

Consequently we find the sum of the series equal to

$$\frac{\Gamma\beta}{\Gamma\alpha\,\Gamma(\beta-\alpha)}\int_0^1 v^{\alpha-1}(1-v)^{\beta-\alpha-1}\varepsilon^{vx}\,dv.$$

Again, if the general term be

$$\frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{\beta(\beta+1)\ldots(\beta+n-1)}\cdot\frac{\alpha'(\alpha'+1)\ldots(\alpha'+n-1)}{\beta'(\beta'+1)\ldots(\beta'+n-1)}\cdot\frac{x^n}{1.2.3\ldots n},$$

the sum of the series will be, in like manner,

$$\frac{\Gamma\beta}{\Gamma\alpha\,\Gamma(\beta-\alpha)},\ \frac{\Gamma\beta'}{\Gamma\alpha'\,\Gamma(\beta'-\alpha')}\ldots\int_0^1\int_0^1\ldots v^{\alpha-1}z^{\alpha'-1}\ldots(1-v)^{\beta-\alpha-1}(1-z)^{\beta'-\alpha'-1}\ldots\varepsilon^{xvz\ldots}\,dv\,dz\ldots$$

To sum the series

$$1+\frac{x}{\beta}+\frac{1}{\beta(\beta+1)}\cdot\frac{x^2}{1.2}+\frac{1}{\beta(\beta+1)(\beta+2)}\cdot\frac{x^3}{1.2.3}+\ldots,$$

$$\frac{1}{\beta(\beta+1)\ldots(\beta+n-1)}=\frac{\Gamma\beta}{\Gamma(\beta+n)}=\frac{\Gamma\beta.\varepsilon}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon^{iz}\,dz}{(1+iz)^{\beta+n}};$$

consequently the required sum is equal to

$$\frac{\Gamma\beta.\varepsilon}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon^{iz}dz}{(1+iz)^{\beta}}\varepsilon^{\frac{x}{1+iz}}\,dz.$$

If it be required to sum the series whose general term is

$$\frac{1}{\beta(\beta+1)\ldots(\beta+n-1)\,\beta'(\beta'+1)\ldots(\beta'+n-1)\ldots}\cdot\frac{x^n}{1.2.3\ldots n},$$

we easily see that the sum is equal to

$$\frac{\Gamma\beta.\varepsilon}{2\pi}\cdot\frac{\Gamma\beta'.\varepsilon}{2\pi}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}dz\,dz'\cdots\frac{\varepsilon^{i(s+sz'+\ldots)}}{(1+iz)^{\beta}(1+iz)^{\beta'}\cdots}\,\varepsilon^{\frac{x}{(1+iz)(1+iz')\cdots}}.$$

We may easily reduce this to a possible form by putting $z=\tan\theta$, $z'=\tan\theta'\ldots$.

If the series to be summed is of the nature of both the kinds of series we have been discussing, we must combine the two methods of summation together; as will be evident in the examples which I shall now give of the application of these series to the solution of differential equations.

For this purpose, I shall avail myself of the methods for expressing the Integrals of Differential Equations by means of series, which are given in the *Philosophical Transactions* for the year 1844.

Let the equation be

$$x^2\frac{d^2u}{dx^2}-\{(a+b-1)x+\mu x^2\}\frac{du}{dx}+\{ab+(e-1)\mu x\}u=0.$$

The symbolical form of this equation is

$$(D-a)(D-b)u-\mu(D-e)\varepsilon^{\theta}u=0;$$

and the solution in series will be

$$u=Ax^a\left\{1+\frac{a-e+1}{a-b+1}\mu x+\frac{(a-e+1)(a-e+2)}{(a-b+1)(a-b+2)}\cdot\frac{\mu^2x^2}{1.2}+\&c.\right\},$$

$$+Bx^b\left\{1+\frac{b-e+1}{b-a+1}\mu x+\frac{(b-e+1)(b-e+2)}{(b-a+1)(b-a+2)}\cdot\frac{\mu^2x^2}{1.2}+\&c.\right\}.$$

Hence we have

$$u=Ax^a\int_0^1 v^{a-e}(1-v)^{e-b-1}\varepsilon^{\mu vx}dv,$$
$$+Bx^b\int_0^1 v^{b-e}(1-v)^{e-a-1}\varepsilon^{\mu vx}dv.$$

I must here remark, that the difference of (a) and (b) is here supposed less than unity. If not, one of these series will require to be modified in a way that will be easily seen. The same remark will apply to several of the equations we are about to consider.

Let the equation be

$$x^3\frac{d^3u}{dx^3}-\{(a+b+c-3)x^2+\mu x^3\}\frac{d^2u}{dx^2}$$
$$+\{(ab+ac+bc-a-b-c+1)x+\mu(e+q-3)x^2\}\frac{du}{dx}$$
$$-\{abc+\mu(eq-c-q+1)x\}u=0.$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)u-\mu(D-e)(D-q)\varepsilon^{\omega}u=0;$$

therefore

$$u=Ax^{a}\left\{1+\frac{(a-e+1)(a-q+1)}{(a-b+1)(a-c+1)}\mu x\right.$$
$$\left.+\frac{(a-e+1)(a-e+2)(a-q+1)(a-q+2)}{(a-b+1)(a-b+2)(a-c+1)(a-c+2)}\cdot\frac{\mu^{2}x^{2}}{1.2}+\&c.\right\}$$
$$+Bx^{b}\left\{1+\frac{(b-e+1)(b-q+1)}{(b-c+1)(b-a+1)}\mu x\right.$$
$$\left.+\frac{(b-e+1)(b-e+2)(b-q+1)(b-q+2)}{(b-c+1)(b-c+2)(b-a+1)(b-a+2)}\cdot\frac{\mu^{2}x^{2}}{1.2}+\&c.\right\}$$
$$+Cx^{c}\left\{1+\frac{(c-e+1)(c-q+1)}{(c-a+1)(c-b+1)}\mu x\right.$$
$$\left.+\frac{(c-e+1)(c-e+2)(c-q+1)(c-q+2)}{(c-a+1)(c-a+2)(c-b+1)(c-b+2)}\cdot\frac{\mu^{2}x^{2}}{1.2}+\&c.\right\}.$$

Hence we shall find

$$u=Ax^{a}\int_{0}^{1}\int_{0}^{1}v^{a-e}z^{a-q}(1-v)^{e-b-1}(1-z)^{q-e-1}\varepsilon^{\mu vzx}\,dv\,dz$$
$$+Bx^{b}\int_{0}^{1}\int_{0}^{1}v^{b-e}z^{b-q}(1-v)^{e-c-1}(1-z)^{q-a-1}\varepsilon^{\mu vzx}\,dv\,dz$$
$$+Cx^{c}\int_{0}^{1}\int_{0}^{1}v^{c-e}z^{c-q}(1-v)^{e-a-1}(1-z)^{q-b-1}\varepsilon^{\mu vzx}\,dv\,dz.$$

Let $x^{3}\dfrac{d^{3}u}{dx^{3}}-(a+b+c-3)x^{2}\dfrac{d^{2}u}{dx^{2}}$

$$+\{(ab+ac+bc-a-b-c+1)x-\mu x^{2}\}\frac{du}{dx}-\{abc-\mu(e-1)x\}u=0.$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)-\mu(D-e)\varepsilon^{\omega}u=0;$$

therefore

$$u=Ax^{a}\left\{1+\frac{a-e+1}{(a-b+1)(a-c+1)}\mu x\right.$$
$$\left.+\frac{(a-e+1)(a-e+2)}{(a-b+1)(a-b+2)(a-c+1)(a-c+2)}\cdot\frac{\mu^{2}x^{2}}{1.2}+\&c.\right\},$$
$$+Bx^{b}\left\{1+\frac{b-e+1}{(b-c+1)(b-a+1)}\mu x\right.$$
$$\left.+\frac{(b-e+1)(b-e+2)}{(b-c+1)(b-c+2)(b-a+1)(b-a+2)}\cdot\frac{\mu^{2}x^{2}}{1.2}+\&c.\right\},$$

$$+ Cx^c \left\{ 1 + \frac{c-e+1}{(c-a+1)(c-b+1)}\mu x + \frac{(c-e+1)(c-e+2)}{(c-a+1)(c-a+2)(c-b+1)(c-b+2)} \cdot \frac{\mu x^2}{1.2} + \ldots \right\}.$$

Consequently

$$u = Ax^a \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, dv\, (1-v)^{f-b-1} v^{e-e} \varepsilon^{\mu v x \cos^2\theta} \cos^{e-c-1}\theta \cos\{\mu v x \cos\theta \sin\theta + (a-c+1)\theta - \tan\theta\}$$

$$+ Bx^b \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, dv\, (1-v)^{f-c-1} v^{b-e} \varepsilon^{\mu v x \cos^2\theta} \cos^{b-a-1}\theta \cos\{\mu v x \cos\theta \sin\theta + (b-a+1)\theta - \tan\theta\}$$

$$+ Cx^c \int_0^1 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} d\theta\, dv\, (1-v)^{f-a-1} v^{c-e} \varepsilon^{\mu v x \cos^2\theta} \cos^{c-b-1}\theta \cos\{\mu v x \cos\theta \sin\theta + (c-b+1)\theta - \tan\theta\}.$$

Let $$x^3 \frac{d^3u}{dx^3} - (a+b+c-3)\, x^2 \frac{d^2u}{dx^2} + (ab+ac+bc-a-b-c+1)\, x \frac{du}{dx} - (abc+\mu x)\, u = 0.$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)\, u - \mu \varepsilon^{\omega} u = 0;$$

therefore

$$u = Ax^a \left\{ 1 + \frac{\mu x^3}{(a-b+1)(a-c+1).1} + \frac{\mu^2 x^2}{(a-b+1)(a-b+2)(a-c+1)(a-c+2).1.2} + \&c. \right\}$$

$$+ Bx^b \left\{ 1 + \frac{\mu x}{(b-c+1)(b-a+1).1} + \frac{\mu^2 x^2}{(b-c+1)(b-c+2)(b-a+1)(b-a+2).1.2} + \&c. \right\}$$

$$+ Cx^c \left\{ 1 + \frac{\mu x}{(c-a+1)(c-b+1).1} + \frac{\mu^2 x^2}{(c-a+1)(c-a+2)(c-b+1)(c-b+2).1.2} + \&c. \right\};$$

$$\therefore\ u = Ax^a \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \varepsilon^{\mu x \cos\theta \cos\phi \cos(\theta+\phi)} \cos^{a-b-1}\theta \cos^{a-c-1}\phi \, d\theta \, d\phi$$

$$\cos\{\mu x \cos\theta \cos\phi \sin(\theta+\phi) + (a-b+1)\theta + (a-c+1)\phi - (\tan\theta + \tan\phi)\}$$

$$+ Bx^b \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \varepsilon^{\mu x \cos\theta \cos\phi \cos(\theta+\phi)} \cos^{b-c-1}\theta \cos^{b-a-1}\phi \, d\theta \, d\phi$$

$$\cos\{\mu x \cos\theta \cos\phi \sin(\theta+\phi) + (b-c+1)\theta + (b-a+1)\phi - (\tan\theta + \tan\phi)\}$$

$$+ Cx^c \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \varepsilon^{\mu x \cos\theta \cos\phi \cos(\theta+\phi)} \cos^{c-a-1}\theta \cos^{c-b-1}\phi \, d\theta \, d\phi$$

$$\cos\{\mu x \cos\theta \cos\phi \sin(\theta+\phi) + (c-a+1)\theta + (c-b+1)\phi - (\tan\theta + \tan\phi)\}.$$

Let $x\dfrac{d^2u}{dx^2} + \dfrac{du}{dx} - \mu u = 0.$

The symbolical form of this equation is

$$D^2u - \mu\varepsilon^{\omega}u = 0,$$

$$u = a_0 + a_1x + a_2x^2 + \ldots + (b_0 + b_1x + b_2x^2 + \ldots)\log_\varepsilon x,$$

where $$a_n = \frac{\mu a_{n-1}}{n^2} - \frac{2\mu b_{n-1}}{n^3}, \quad b_n = \frac{\mu b_{n-1}}{n^2}.$$

It will be necessary to discuss the first only of these series. We find

$$a_n = \frac{\mu^n a_0}{1^2.2^2.3^2\ldots n^2} - \frac{2\mu^n b_0}{1^2.2^2.3^2\ldots n^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots \frac{1}{n}\right)$$

$$= \frac{\mu^n a_0}{1^2.2^2.3^2\ldots n^2} - \frac{2\mu^n b_0}{1^2.2^2.3^2\ldots n^2}\int_0^1 \frac{(v^n - 1)\,dv}{v-1},$$

and consequently we shall find

$$a_0 + a_1x + a_2x^2 + \ldots = a_0\left(1 + \frac{\mu x}{1^2} + \frac{\mu^2x^2}{1^2.2^2} + \frac{\mu^3x^3}{1^2.2^2.3^2} + \ldots\right)$$

$$-2b_0\int_0^1 \frac{dv}{v-1}\left\{\left(1 + \frac{\mu v x}{1^2} + \frac{\mu^2v^2x^2}{1^2.2^2} + \ldots\right) - \left(1 + \frac{\mu x}{1^2} + \frac{\mu^2x^2}{1^2.2^2} + \ldots\right)\right\},$$

which reduces the series to forms we have already considered.

Let $x^3\dfrac{d^3u}{dx^3} + 3x^2\dfrac{d^2u}{dx^2} + x\dfrac{du}{dx} - \mu x u = 0.$

The symbolical form of this equation is

$$D^3u - \mu\varepsilon^{\omega}u = 0,$$

$$u = (a_0 + a_1x + a_2x^2 + \ldots) + \log_\varepsilon x(b_0 + b_1x + b_2x^2 + \ldots) + (\log_\varepsilon x)^2(c_0 + c_1x + c_2x^2 + \ldots),$$

where

$$a_n = \frac{\mu(n^2a_{n-1} - 3nb_{n-1} + 12c_{n-1})}{n^5}, \quad b_n = \frac{\mu(nb_{n-1} - 6c_{n-1})}{n^4},$$

$$c_n = \frac{\mu c_{n-1}}{n^3}.$$

I shall discuss the first only of these series. We shall find

$$a_n = \frac{a_0\mu^n}{1^3.2^3.3^3\ldots n^3} - \frac{3\mu^n b_0}{1^3.2^3.3^3\ldots n^3}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$$
$$+ \frac{9\mu^n c_0}{1^3.2^3.3^3\ldots n^3}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)^2$$
$$+ \frac{3\mu^n c_0}{1^3.2^3.3^3\ldots n^3}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \ldots + \frac{1}{n^2}\right)$$
$$= \frac{a_0\mu^n}{1^3.2^3.3^3\ldots n^3} - \frac{3\mu^n b_0}{1^3.2^3.3^3\ldots n^3}\int_0^1 \frac{dv(v^n - 1)}{v - 1}$$
$$+ \frac{9\mu^n c_0}{1^3.2^3.3^3\ldots n^3}\int_0^1\int_0^1 dv\,dz\,\frac{v^n - 1}{v - 1}\cdot\frac{z^n - 1}{z - 1},$$
$$+ \frac{3\mu^n c_0}{1^3.2^3.3^3\ldots n^3}\int_0^1\int_0^1 dv\,dz\,\frac{v^nz^n - 1}{vz - 1}$$

and proceed as in the last example.

Let $$x^3\frac{d^3u}{dx^3} - (2a + b - 3)x^2\frac{d^2u}{dx^2} + (a^2 + 2ab - 2a - b + 1)x\frac{du}{dx}$$
$$- (a^2b + \mu x)u = 0.$$

The symbolical form of this equation is

$$(D - a)^2(D - b)u - \mu\varepsilon^{\omega}u = 0.$$

The solution will be found to involve the series

$$A_0 + A_1x + A_2x^2 + \ldots\ldots$$

where

$$A_n = \frac{\mu^n A_0}{1^2\ldots n^2(1 + a - b)\ldots(n + a - b)} - \frac{\mu^n B_0}{1^2\ldots n^2(1 + a - b)\ldots(n + a - b)}$$
$$\left\{\frac{3.1 + 2(a - b)}{1(1 + a - b)} + \frac{3.2 + 2(a - b)}{2(2 + a - b)} + \ldots + \frac{3n + 2(a - b)}{n(n + a - b)}\right\}.$$

Now $$\frac{3}{1+a-b}+\frac{3}{2+a-b}+\frac{3}{3+a-b}+\ldots+\frac{3}{n+a-b}$$

$$=3\int_0^1 \frac{v^{n-b}dv(v^n-1)}{v-1}$$

$$\frac{2(a-b)}{1(1+a-b)}+\frac{2(a-b)}{2(2+a-b)}+\frac{2(a-b)}{3(3+a-b)}\ldots+\frac{2(a-b)}{n(n+a-b)}$$

$$=2(a-b)\int_{0^n}^1\int_0^1 dv\,dz\,\frac{(v^nz^n-1)\,z^{n-b}}{vz-1},$$

whence the rest is obvious.

The preceding equations are all included under the general symbolical form

$$u+\phi(D)\,\varepsilon^{rw}u=0,$$

to which the still more general symbolical form

$$u+a_1\phi(D)\,\varepsilon^{w}u+a_2\phi(D)\,\phi(D-1)\,\varepsilon^{2w}u+\&c.=0$$

can always be reduced.

I shall conclude this paper with two examples, which are well calculated to exhibit the power of the method I have been investigating.

Let $\dfrac{d^ny}{dx^n}=x^my.$

$$y=A_0\left\{1+\frac{x^{m+n}}{(m+1)\ldots(n+m)}+\frac{x^{2(n+m)}}{(m+1)\ldots(n+m)(n+2m+1)\ldots(2n+2m)}\right.$$
$$\left.+\frac{x^{3(n+m)}}{(m+1)\ldots(n+m)(n+2m+1)\ldots(2n+2m)(2n+3m+1)\ldots3(n+m)}+\&c.\right\}$$
$$+A_1x\left\{1+\frac{x^{m+n}}{(m+2)\ldots(n+m+1)}+\frac{x^{2(m+n)}}{(m+2)\ldots(n+m+1)(n+2m+2)\ldots(2n+2m+1)}\right.$$
$$\left.+\frac{x^{3(m+n)}}{(m+2)\ldots(n+m+1)(n+2m+2)\ldots(2n+2m+1)(2n+3m+2)\ldots(3n+3m+1)}+\&c.\right\}$$
$$+A_2x^2\left\{1+\frac{x^{m+n}}{(m+3)\ldots(n+m+2)}+\frac{x^{2(m+n)}}{(m+3)\ldots(n+m+2)(n+2m+3)\ldots(2n+2m+2)}\right.$$
$$\left.+\frac{x^{3(n+m)}}{(m+3)\ldots(n+m+2)(n+2m+3)\ldots(2n+2m+2)(2n+3m+3)\ldots(3n+3m+2)}+\&c.\right\}$$
$$+\&c.$$

$$= A_0 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\varepsilon^{i(z+z'+\ldots z^{(n-2)})}\, dz\, dz' \ldots dz^{(n-2)}}{(1+iz)^{\frac{m+1}{n+m}}\,(1+iz')^{\frac{m+2}{n+m}}\ldots(1+iz^{(n-2)})^{\frac{n+m-1}{n+m}}}$$

$$\varepsilon^{\frac{x^{n+m}}{(n+m)^n(1+iz)(1+iz')\ldots}}$$

$$+ A_1 x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots \frac{\varepsilon^{i(z+z'+\ldots z^{(n-2)})}\, dz\, dz' \ldots dz^{(n-2)}}{(1+iz)^{\frac{m+2}{n+m}}\,(1+iz')^{\frac{m+3}{n+m}}\ldots(1+iz^{(n-2)})^{\frac{n+m+1}{n+m}}}$$

$$\varepsilon^{\frac{x^{n+m}}{(n+m)^n(1+iz)(1+iz')\ldots}}$$

$$+ A_2 x^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \ldots \frac{\varepsilon^{i(z+z'+\ldots z^{(n-2)})}\, dz\, dz' \ldots dz^{(n-2)}}{(1+iz)^{\frac{m+3}{n+m}}\,(1+iz')^{\frac{m+4}{n+m}}\ldots(1+iz^{(n-2)})^{\frac{n+m+2}{n+m}}}$$

$$\varepsilon^{\frac{x^{n+m}}{(n+m)^n(1+iz)(1+iz')\ldots}}$$

+ &c.,

where the integral may be transformed by putting

$$z = \tan\theta, \quad z' = \tan\theta'.$$

Let $x\dfrac{d^n y}{dx^n} + \dfrac{d^{n-1}y}{dx^{n-1}} - y = 0.$

$$y = y_0\left\{1 + \frac{x^{n-1}}{1.2.3\ldots(n-1).1} + \frac{x^{2n-2}}{1.2.3\ldots(2n-2).1.n} + \frac{x^{3n-3}}{1.2.3\ldots(3n-3).1.n.(2n-1)} + \ldots\right\}$$

$$+ y_0'\left\{x + \frac{x^n}{1.2.3\ldots n.2} + \frac{x^{2n-1}}{1.2.3\ldots(2n-1).2.(n+1)} + \frac{x^{3n-2}}{1.2.3\ldots(3n-2).2(n+1).2n} + \ldots\right\}$$

$$+ y_0''\left\{\frac{x^2}{1.2} + \frac{x^{n+1}}{1.2.3\ldots(n+1).3} + \frac{x^{2n}}{1.2.3\ldots 2n.3.(n+2)} + \frac{x^{3n-1}}{1.2.3\ldots(3n-1).3.(n+2)\,(2n+1)} + \ldots\right\}$$

+ &c. to $(n-1)$ terms,

$$= E_0 \int_{-\infty}^{\infty} \frac{\varepsilon^{iz} dz}{(1+iz)^{\frac{1}{n-1}}} \left\{ \varepsilon^{\frac{x}{\rho}} + \alpha^{n-1} \varepsilon^{\frac{ax}{\rho}} + \alpha^{2(n-1)} \varepsilon^{\frac{a^2x}{\rho}} + \&c. \right\}$$

$$+ E_1 \int_{-\infty}^{\infty} \frac{\varepsilon^{iz} dz}{(1+iz)^{\frac{1}{n-1}}} \left\{ \varepsilon^{\frac{x}{\rho}} + \alpha^{n-2} \varepsilon^{\frac{ax}{\rho}} + \alpha^{2(n-2)} \varepsilon^{\frac{a^2x}{\rho}} + \&c. \right\}$$

$$+ E_2 \int_{-\infty}^{\infty} \frac{\varepsilon^{iz} dz}{(1+iz)^{\frac{1}{n-1}}} \left\{ \varepsilon^{\frac{x}{\rho}} + \alpha^{n-3} \varepsilon^{\frac{ax}{\rho}} + \alpha^{2(n-3)} \varepsilon^{\frac{a^2x}{\rho}} + \&c. \right\}$$

$$+ \&c.,$$

where $\rho = (n-1)^{\frac{1}{n-1}} (1+iz)^{\frac{1}{n-1}}$; $1, \alpha, \alpha^2, \ldots$ are the $(n-1)$ roots of the equation $\xi^{n-1} - 1 = 0$, and each of the series contained within the brackets is continued to $(n-1)$ terms. We may transform this expression by putting $z = \tan\theta$. By the above process we shall obtain $(n-1)$ particular integrals only; but when $(n-1)$ particular integrals of a linear equation of the n^{th} order are known, the remaining integral can always be found.

ON THE INTEGRATION OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

By W. H. L. Russell.

In a paper which appears in the present number of this *Journal*, I gave a new method for expressing the sums of certain infinite series by means of definite integrals, and applied the results to the integration of a large class of linear differential equations. I now propose to investigate a similar method for the integration of a class of linear partial differential equations. If we express the solutions of these equations by means of series, we shall find in the cases which it is the object of this paper to consider, that these series are analogous to those which were discussed in the paper just mentioned, but that the terms are successively multiplied by the differential coefficients of an arbitrary function, the indices of differentiation receiving a constant increment each term. Now an arbitrary function $F(x)$ can sometimes be expressed by the integral

$$\int_0^\rho \varpi(\rho)\, \varepsilon^{\rho x}\, d\rho,$$

and always by the integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon^{i\alpha(x-x')}\, F(x')\,\frac{d\alpha.dx'}{2\pi};$$

consequently, in both cases, the quantities under the integral sign in the successive differential coefficients of these integrals with respect to (x) are in geometrical progression. Hence the substitution of these integrals reduces the proposed series to forms already considered. In this paper I intend to use the first of these expressions, as it is the simpler of the two; the application of the second is to be conducted in a similar manner. The method of summation I have employed in this paper, will be understood on comparing it with the paper to which I have already referred.

Let $x_1^3\frac{d^3u}{dx_1^3} - (a+b+c-3)\,x_1^2\frac{d^2u}{dx_1^2}$

$$+ (bc+ca+ab-a-b-c+1)\,x_1\frac{du}{dx_1} - abcu = x_1\frac{du}{dx_2}.$$

The symbolical form of this equation is

$$(D-a)(D-b)(D-c)\,u - \frac{d}{dx_2}\varepsilon^{\omega}u = 0.$$

Hence,

$$u = x_1^a\left\{1 + \frac{x_1}{(a-b+1)(a-c+1).1}\frac{d}{dx_2}\right.$$

$$\left. + \frac{x_1^2}{(a-b+1)(a-b+2)(a-c+1)(a-c+2).1.2}\frac{d^2}{dx_2^2} + \&c.\right\} F_a x_2,$$

$$+ x_1^b\left\{1 + \frac{x_1}{(b-c+1)(b-a+1).1}\frac{d}{dx_2}\right.$$

$$\left. + \frac{x_1^2}{(b-c+1)(b-c+2)(b-a+1)(b-a+2).1.2}\frac{d^2}{dx_2^2} + \&c.\right\} F_b x_2,$$

$$+ x_1^c\left\{1 + \frac{x_1}{(c-a+1)(c-b+1).1}\frac{d}{dx_2}\right.$$

$$\left. + \frac{x_1^2}{(c-a+1)(c-a+2)(c-b+1)(c-b+2).1.2}\frac{d^2}{dx_2^2} + \&c.\right\} F_c x_2.$$

Let $F_a x_2 = \int_0^{\rho} \varpi_a(\rho)\,\varepsilon^{\rho x_2}\,d\rho$, and similarly for $F_b x_2$, $F_c x_2$,

$$u = x^a\int_0^{\rho}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \varepsilon^{\rho(x_1\cos\theta\cos\phi\cos(\theta+\phi)+x_2)}\cos^{a-b-1}\theta\cos^{a-c-1}\phi\,\varpi_a(\rho)\,d\theta\,d\phi\,d\rho$$

$$\cos\{\rho x_1\cos\theta\cos\phi\sin(\theta+\phi)+(a-b+1)\theta+(a-c+1)\phi-(\tan\theta+\tan\phi)\},$$

$$+x^b\int_0^\rho\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\varepsilon^{\rho(x_1\cos\theta\cos\phi\cos(\theta+\phi)+x_2)}\cos^{b-c-1}\theta\cos^{b-a-1}\phi\varpi_b(\rho)d\theta d\phi d\rho$$

$$\cos\{\rho x_1\cos\theta\cos\phi\sin(\theta+\phi)+(b-c+1)\theta+(b-a+1)\phi-(\tan\theta+\tan\phi)\},$$

$$+x^c\int_0^\rho\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\varepsilon^{\rho(x_1\cos\theta\cos\phi\cos(\theta+\phi)+x_2)}\cos^{c-a-1}\theta\cos^{c-b-1}\phi\varpi_c(\rho)d\theta d\phi d\rho$$

$$\cos\{\rho x_1\cos\theta\cos\phi\sin(\theta+\phi)+(c-a+1)\theta+(c-b+1)\phi-(\tan\theta+\tan\phi)\}.$$

Let $$x_1\frac{d^2u}{dx_1^2}+\frac{du}{dx_1}-\frac{du}{dx_2}=0.$$

The symbolical form of this equation is

$$D^2u-\frac{d}{dx_2}\varepsilon^\omega u=0,$$

$$u=F_0x_2+x_1F_1x_2+x_1^2F_2x_2+\&c.+\log_\varepsilon x_1(f_0x_2+x_1f_1x_2+x_1^2f_2x_2+\&c.)$$

where $$F_nx_2=\frac{d}{dx_2}\cdot\frac{F_{n-1}x_2}{n^2}-2\frac{d}{dx_2}\cdot\frac{f_{n-1}x_2}{n^2},\qquad f_nx_2=\frac{d}{dx_2}\frac{f_{n-1}x_2}{n^2}.$$

I shall discuss the first only of these series,

$$F_nx_2=\frac{d^n}{dx_2^n}\frac{F_0x_2}{1^2.2^2.3^2\ldots n^2}-2\frac{d^n}{dx_2^n}\frac{f_0x_2}{1^2.2^2.3^2\ldots n^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right).$$

Let $F_0x_2=\int_0^\rho\Pi(\rho)\varepsilon^{\rho x_2}d\rho$, and similarly for f_0x_2; then

$$F_nx_2=\int_0^\rho\frac{\Pi(\rho)\varepsilon^{\rho x_2}\rho^n d\rho}{1^2.2^2.3^2\ldots n^2}$$

$$-2\int_0^\rho\frac{\varpi(\rho)\varepsilon^{\rho x_2}\rho^n d\rho}{1^2.2^2.3^2\ldots n^2}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right),$$

a form which I have considered in the paper I have before mentioned.

Let $$\frac{d^vu}{dx_1dx_2dx_3\ldots dx_v}=u.$$

$$u=\int dx_vF_vx_v+\frac{x_1x_2x_3\ldots x_{v-1}}{1^{v-1}}\iint dx_v^2F_vx_v$$

$$+\frac{x_1^2x_2^2x_3^2\ldots x_{v-1}^2}{1^{v-1}.2^{v-1}}\iiint dx_v^3F_vx_v+\&c.,$$

$$+\int dx_{v-1}F_{v-1}x_{v-1}+\frac{x_1x_2x_3\ldots x_v}{1^{v-1}}\iint dx_{v-1}^2F_{v-1}x_{v-1}$$

$$+\frac{x_1^2x_2^2x_3^2\ldots x_v^2}{1^{v-1}.2^{v-1}}\iiint dx_{v-1}^3F_{v-1}x_{v-1}+\&c.,$$

$$+\int dx_{v-2}F_{v-2}x_{v-2}+\frac{x_1x_2x_3\ldots x_v}{1^{v-1}}\iint dx^2_{v-2}F_{v-2}x_{v-2}$$

$$+\frac{x_1^3x_2^3x_3^3\ldots x_v^3}{1^{v-1}.2^{v-1}}\iiint dx^3_{v-2}F_{v-2}x_{v-2}+\&c.,$$

+ &c.

Now $\int_0\int_0\int_0\ldots dx^nFx=\dfrac{1}{1.2.3\ldots(n-1)}\int_0^x(x-\rho)^{n-1}F(\rho)d\rho$;

therefore

$$u=\int_0^{x_v}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots F_v\rho\,\frac{\varepsilon^{i(z+z'+\ldots z^{(v-2)})}dz\,dz'\ldots dz^{(v-2)}d\rho}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})}$$

$$\varepsilon^{\frac{x_1x_2x_3\ldots x_{v-1}(x_v-\rho)}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})}}$$

$$+\int_0^{x_{v-1}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots F_{v-1}\rho\,\frac{\varepsilon^{i(z+z'+\ldots z^{(v-2)})}dz\,dz'\ldots dz^{(v-2)}d\rho}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})}$$

$$\varepsilon^{\frac{x_1x_2x_3\ldots(x_{v-1}-\rho)\,x_v}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})}}$$

$$+\int_0^{x_{v-2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\ldots F_{v-2}\rho\,\frac{\varepsilon^{i(z+z'+\ldots z^{(v-2)})}dz\,dz'\ldots dz^{(v-2)}d\rho}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})}$$

$$\varepsilon^{\frac{x_1x_2x_3\ldots(x_{v-2}-\rho)\ldots x_v}{(1+iz)(1+iz')\ldots(1+iz^{(v-2)})}}$$

+ &c.,

where the integral may be transformed by putting $z=\tan\theta$, $z'=\tan\theta'$, &c.

SOME THEOREMS ON CONFOCAL SURFACES OF THE SECOND ORDER.

By the Rev. J. Booth, F.R.S., &c.

The following demonstrations of some general theorems on Confocal Surfaces of the Second Order may be acceptable to such readers of the Journal as take an interest in researches of this kind.

Theorem I. *Three confocal surfaces of the second order intersect in a common point Q the vertex of a cone which envelopes a fourth confocal surface; to determine the equation*

of this cone referred to the normals of the three surfaces, at the common point Q, as axes of coordinates.

It is very generally known that confocal surfaces mutually intersect at right angles;* the normals of the three surfaces will therefore constitute a system of rectangular coordinates.

Let the equation of the fourth surface, an ellipsoid suppose, be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and let α, β, γ be the coordinates of the point Q, the vertex of the enveloping cone.

Let $\quad x-\alpha=m(z-\gamma), \quad y-\beta=n(z-\gamma) \ldots\ldots\ldots (2),$

be the equations of a right line passing through the point $(\alpha\beta\gamma)$, and piercing the surface (1) in two points. If we substitute the values of x and y derived from (2) in (1), we shall have a resulting equation of the form

$$Uz^2+2Vz+W=0 \ldots\ldots\ldots\ldots\ldots (3);$$

of which the two roots will express the values of the vertical ordinates of the two points in which the line pierces the surface. When the line becomes a side of the enveloping cone, the two roots become equal, and we get the well-known condition

$$UW-V^2=0 \ldots\ldots\ldots\ldots\ldots (4).$$

If we substitute the values of x and y derived from (2) in (1), we shall have

$$U=\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{1}{c^2}\right],\quad W=\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right],\quad V=\left[\frac{m\alpha}{a^2}+\frac{n\beta}{b^2}+\frac{\gamma}{c^2}\right]$$
$$\ldots\ldots\ldots (5),$$

and by substitution the equation of condition (4) will become

$$\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{1}{c^2}\right]\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]=\left[\frac{m\alpha}{a^2}+\frac{n\beta}{b^2}+\frac{\gamma}{c^2}\right]^2 \ldots (6);$$

or, putting for m and n their values

$$m=\frac{x-\alpha}{z-\gamma},\quad n=\frac{y-\beta}{z-\gamma} \ldots\ldots\ldots\ldots (7),$$

* See Dupin's *Développements de Géométrie.*

the equation of the enveloping cone becomes

$$\left.\begin{aligned}\left[\frac{(x-\alpha)^2}{a^2}+\frac{(y-\beta)^2}{b^2}+\frac{(z-\gamma)^2}{c^2}\right]\left[\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}-1\right]\\ =\left[\frac{\alpha(x-\alpha)}{a^2}+\frac{\beta(y-\beta)}{b^2}+\frac{\gamma(z-\gamma)}{c^2}\right]^2\end{aligned}\right\}\ldots(8).$$

We shall leave the equation of the enveloping cone in this form, to facilitate its future transformation.

Let the tangent planes at the common intersection Q of the three confocal surfaces be taken for the coordinate planes, to which the equation of the enveloping cone is to be referred, and let $a_i b_i c_i$, $a_{ii} b_{ii} c_{ii}$, $a_{iii} b_{iii} c_{iii}$, be the semi-axes of the three confocal surfaces. Moreover, let p, p_i, p_{ii} be the perpendiculars from the common centre of the surfaces on the three tangent planes at the point Q. Now the normals, at the point Q, to the three surfaces being parallel to the perpendiculars p, p_i, p_{ii}, make the same angles with the original axes of coordinates.

Let $\lambda\mu\nu$, $\lambda'\mu'\nu'$, $\lambda''\mu''\nu''$, be the angles which the perpendiculars p, p_i, p_{ii} respectively make with the axes of coordinates. Then as each abscissa is equal to the projections of the three new ones upon it,

$$\left.\begin{aligned}x-\alpha &= x'\cos\lambda + y'\cos\lambda' + z'\cos\lambda'',\\ y-\beta &= x'\cos\mu + y'\cos\mu' + z'\cos\mu'',\\ z-\gamma &= x'\cos\nu + y'\cos\nu' + z'\cos\nu'',\end{aligned}\right\}\ldots\ldots(9),$$

but $$\cos\lambda = \frac{p\alpha}{a_i^2},\quad \cos\lambda' = \frac{p_i\alpha}{a_{ii}^2},\quad \cos\lambda'' = \frac{p_{ii}\alpha}{a_{iii}^2}\ldots\ldots(10);$$

finding similar values for μ, μ', μ'' and ν, ν', ν'', then substituting the resulting values in (9), we shall have

$$\left.\begin{aligned}x-\alpha &= \alpha\left[\frac{px'}{a_i^2}+\frac{p_iy'}{a_{ii}^2}+\frac{p_{ii}z'}{a_{iii}^2}\right]\\ y-\beta &= \beta\left[\frac{px'}{b_i^2}+\frac{p_iy'}{b_{ii}^2}+\frac{p_{ii}z'}{b_{iii}^2}\right]\\ z-\gamma &= \gamma\left[\frac{px'}{c_i^2}+\frac{p_iy'}{c_{ii}^2}+\frac{p_{ii}z'}{c_{iii}^2}\right]\end{aligned}\right\}\ldots\ldots\ldots(11).$$

Substituting these values in (8), the equation of the enveloping cone, and eliminating the equivalent terms, we shall have, after some reductions,

$$\left.\begin{aligned}
&\left[\left(\frac{\alpha\beta}{ab}\right)^2+\left(\frac{\alpha\gamma}{ac}\right)^2-\left(\frac{\alpha}{a}\right)^2\right]\left[\frac{px}{a_{,}^2}+\frac{p_{,}y}{a_{,,}^2}+\frac{p_{,,}z}{a_{,,,}^2}\right]^2\\
&+\left[\left(\frac{\beta\gamma}{bc}\right)^2+\left(\frac{\beta\alpha}{ba}\right)^2-\left(\frac{\beta}{b}\right)^2\right]\left[\frac{px}{b_{,}^2}+\frac{p_{,}y}{b_{,,}^2}+\frac{p_{,,}z}{b_{,,,}^2}\right]^2\\
&+\left[\left(\frac{\gamma\alpha}{ca}\right)^2+\left(\frac{\gamma\beta}{cb}\right)^2-\left(\frac{\gamma}{c}\right)^2\right]\left[\frac{px}{c_{,}^2}+\frac{p_{,}y}{c_{,,}^2}+\frac{p_{,,}z}{c_{,,,}^2}\right]^2\\
&+2\left(\frac{\beta\gamma}{bc}\right)^2\left[\frac{px}{b_{,}^2}+\frac{p_{,}y}{b_{,,}^2}+\frac{p_{,,}z}{b_{,,,}^2}\right]\left[\frac{px}{c_{,}^2}+\frac{p_{,}y}{c_{,,}^2}+\frac{p_{,,}z}{c_{,,,}^2}\right]\\
&+2\left(\frac{\gamma\alpha}{ca}\right)^2\left[\frac{px}{c_{,}^2}+\frac{p_{,}y}{c_{,,}^2}+\frac{p_{,,}z}{c_{,,,}^2}\right]\left[\frac{px}{a_{,}^2}+\frac{p_{,}y}{a_{,,}^2}+\frac{p_{,,}z}{a_{,,,}^2}\right]\\
&=2\left(\frac{\alpha\beta}{ab}\right)^2\left[\frac{px}{a_{,}^2}+\frac{p_{,}y}{a_{,,}^2}+\frac{p_{,,}z}{a_{,,,}^2}\right]\left[\frac{px}{b_{,}^2}+\frac{p_{,}y}{b_{,,}^2}+\frac{p_{,,}z}{b_{,,,}^2}\right]
\end{aligned}\right\}\quad\ldots(12),$$

omitting the traits over the xyz as no longer necessary.

Now, as the surfaces are confocal, let

$$\left.\begin{aligned}
a_{,}^2&=a^2+k^2, & b_{,}^2&=b^2+k^2, & c_{,}^2&=c^2+k^2\\
a_{,,}^2&=a^2+k_{,}^2, & b_{,,}^2&=b^2+k_{,}^2, & c_{,,}^2&=c^2+k_{,}^2\\
a_{,,,}^2&=a^2+k_{,,}^2, & b_{,,,}^2&=b^2+k_{,,}^2, & c_{,,,}^2&=c^2+k_{,,}^2
\end{aligned}\right\}\quad\ldots(13);$$

and as $\alpha\beta\gamma$ is a point on each of the three confocal surfaces, we shall have

$$\left.\begin{aligned}
\frac{\alpha^2}{a^2+k^2}+\frac{\beta^2}{b^2+k^2}+\frac{\gamma^2}{c^2+k^2}&=1\\
\frac{\alpha^2}{a^2+k_{,}^2}+\frac{\beta^2}{b^2+k_{,}^2}+\frac{\gamma^2}{c^2+k_{,}^2}&=1\\
\frac{\alpha^2}{a^2+k_{,,}^2}+\frac{\beta^2}{b^2+k_{,,}^2}+\frac{\gamma^2}{c^2+k_{,,}^2}&=1
\end{aligned}\right\}\quad\ldots\ldots(14),$$

whence

$$\left.\begin{aligned}
\alpha^2&=\frac{(a^2+k^2)(a^2+k_{,}^2)(a^2+k_{,,}^2)}{(a^2-c^2)(a^2-b^2)}\\
\beta^2&=\frac{(b^2+k^2)(b^2+k_{,}^2)(b^2+k_{,,}^2)}{(b^2-c^2)(b^2-a^2)}\\
\gamma^2&=\frac{(c^2+k^2)(c^2+k_{,}^2)(c^2+k_{,,}^2)}{(c^2-a^2)(c^2-b^2)}
\end{aligned}\right\}\quad\ldots\ldots(15).$$

If we now perform the operations of multiplication, indicated in the equation (12) of the enveloping cone, we shall

find that the coefficient of the term xy will be as follows:

$$2pp_{,}\left[\left(\frac{\beta\gamma(b^2-c^2)}{bc\,b_{,}c_{,}\,b_{,,}c_{,,}}\right)^2+\left(\frac{\gamma\alpha(c^2-a^2)}{ca\,c_{,}a_{,}\,c_{,,}a_{,,}}\right)^2+\left(\frac{\alpha\beta(a^2-b^2)}{ab\,a_{,}b_{,}\,a_{,,}b_{,,}}\right)^2\right]$$

$$-2pp_{,}\left[\frac{\alpha^2}{a^2a_{,}^2a_{,,}^2}+\frac{\beta^2}{b^2b_{,}^2b_{,,}^2}+\frac{\gamma^2}{c^2c_{,}^2c_{,,}^2}\right]\ldots\ldots(16).$$

Now if we eliminate from this expression the quantities α, β, γ, by the help of (15), the first term of the preceding expression will become

$$\left[\frac{a^2(b^2-c^2)(b^2+k_{,,}^2)(c^2+k_{,,}^2)+b^2(c^2-a^2)(a^2+k_{,,}^2)(c^2+k_{,,}^2)+c^2(a^2-b^2)(a^2+k_{,,}^2)(b^2+k_{,,}^2)}{a^2b^2c^2(b^2-c^2)(c^2-a^2)(a^2-b^2)}\right];$$

and this expression may be reduced to

$$\left[\frac{a^2(b^4-c^4)+b^2(c^4-a^4)+c^2(a^4-b^4)}{a^2b^2c^2(b^2-c^2)(c^2-a^2)(a^2-b^2)}\right]k_{,,}^2.$$

Now the numerator of this fraction may be written in the form $(b^2-c^2)(c^2-a^2)(a^2-b^2)$. Hence the first member of the coefficient of xy in (16) may be reduced to $\frac{2pp_{,}k_{,,}^2}{a^2b^2c^2}$.

If in the same manner we calculate the second member of (16), we shall find it to be $\frac{-2pp_{,}k_{,,}^2}{a^2b^2c^2}$. Hence the coefficient of xy is 0. In the same way it may be shown that the coefficients of xz and yz are each = 0.

We have now to determine the coefficient of x^2 in the equation (12) of the cone.

Collecting the terms by which x^2 is multiplied in that equation, we shall find their sum to be

$$p^2\left[\left(\frac{\beta\gamma}{bc_{,}}\right)^2\left(\frac{1}{b_{,}^2}-\frac{1}{c_{,}^2}\right)^2+\left(\frac{\gamma\alpha}{ca}\right)^2\left(\frac{1}{c_{,}^2}-\frac{1}{a_{,}^2}\right)^2+\left(\frac{\alpha\beta}{ab}\right)^2\left(\frac{1}{a_{,}^2}-\frac{1}{b_{,}^2}\right)^2\right]$$

$$-p^2\left[\frac{\alpha^2}{a^2a_{,}^4}+\frac{\beta^2}{b^2b_{,}^4}+\frac{\gamma^2}{c^2c_{,}^4}\right]\ldots\ldots(17).$$

If, in the first member of this expression, we substitute for $\alpha\beta\gamma$ their values derived from (15), the transformed result will become

$$p^2\begin{vmatrix}+a^2(b^2-c^2)(b^2+k_{,}^2)(b^2+k_{,,}^2)(c^2+k_{,}^2)(c^2+k_{,,}^2)(a^2+k^2)\\+b^2(c^2-a^2)(c^2+k_{,}^2)(c^2+k_{,,}^2)(a^2+k_{,}^2)(a^2+k_{,,}^2)(b^2+k^2)\\+c^2(a^2-b^2)(a^2+k_{,}^2)(a^2+k_{,,}^2)(b^2+k_{,}^2)(b^2+k_{,,}^2)(c^2+k^2)\end{vmatrix}(\ldots18),$$

divided by the denominator

$$a^2b^2c^2a_i^2b_i^2c_i^2(b^2-c^2)(c^2-a^2)(a^2-b^2).$$

Making like substitutions in the second member of (17), the resulting expression will become

$$-p^2\begin{Vmatrix} b^2c^2(b^2-c^2)(a^2+k_i^2)(a^2+k_{ii}^2)(b^2+k^2)(c^2+k^2) \\ +c^2a^2(c^2-a^2)(b^2+k_i^2)(b^2+k_{ii}^2)(a^2+k^2)(c^2+k^2) \\ +a^2b^2(a^2-b^2)(c^2+k_i^2)(c^2+k_{ii}^2)(a^2+k^2)(b^2+k^2) \end{Vmatrix} \ldots(19),$$

divided by the same denominator as the preceding expression.

If we add together the terms in (18) that are multiplied by a^2c^2, we shall find the result to be

$$-a^2c^2(a^2-c^2)(b^2+k_i^2)(b^2+k_{ii}^2)[a^2c^2+(a^2+c^2)k^2+k^2(k_i^2+k_{ii}^2)-k_i^2k_{ii}^2],$$

and the corresponding term in (19) is

$$a^2c^2(a^2-c^2)(b^2+k_i^2)(b^2+k_{ii}^2)[a^2c^2+k^2(a^2+c^2)+k^4];$$

adding these expressions together, the resulting expression will become

$$a^2c^2(a^2-c^2)(b^2+k_i^2)(b^2+k_{ii}^2)(k^2-k_i^2)(k^2-k_{ii}^2)\ldots(19);$$

or, developing this expression,

$$[a^4b^4c^2+a^4b^2c^2(k_i^2+k_{ii}^2)+a^4c^2k_i^2k_{ii}^2](k^2-k_i^2)(k^2-k_{ii}^2)$$
$$-[a^2c^4b^4+a^2b^2c^4(k_i^2+k_{ii}^2)+a^2c^4k_i^2k_{ii}^2](k^2-k_i^2)(k^2-k_{ii}^2).$$

Hence, if in like manner we develope and combine the remaining terms in (18) and (19), the whole coefficient of x^2 will be the sum of the following twenty-four terms,

$$\begin{Bmatrix} a^4b^4c^2+a^4b^2c^2(k_i^2+k_{ii}^2)+a^4c^2k_i^2k_{ii}^2 \\ -a^2b^4c^4-a^2b^2c^4(k_i^2+k_{ii}^2)-a^2c^4k_i^2k_{ii}^2 \\ +b^4c^4a^2+b^4c^2a^2(k_i^2+k_{ii}^2)+b^4a^2k_i^2k_{ii}^2 \\ -b^2c^4a^4-b^2c^2a^4(k_i^2+k_{ii}^2)-b^2a^4k_i^2k_{ii}^2 \\ +c^4a^4b^2+c^4a^2b^2(k_i^2+k_{ii}^2)+c^4b^2k_i^2k_{ii}^2 \\ -c^2a^4b^4-c^2a^2b^4(k_i^2+k_{ii}^2)-c^2b^4k_i^2k_{ii}^2 \end{Bmatrix} p^2(k^2-k_i^2)(k^2-k_{ii}^2)\ldots(20),$$

divided by the denominator

$$-a^2b^2c^2a_i^2b_i^2c_i^2(b^2-c^2)(c^2-a^2)(a^2-b^2).$$

Now if we add the terms of this coefficient vertically, the sum of the first column = 0, the sum of the second is also = 0,

and the sum of the terms in the third column may be reduced to

$$(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)\, k_{,}^{\,2} k_{,,}^{\,2}.$$

Hence the coefficient of x^2 becomes

$$\frac{p^2 k^2 k_{,}^{\,2} k_{,,}^{\,2} (k^2 - k_{,}^{\,2})(k^2 - k_{,,}^{\,2})}{k^2 a_{,}^{\,2} b_{,}^{\,2} c_{,}^{\,2} a^2 b^2 c^2} \quad \ldots\ldots\ldots\ldots (21).$$

The symmetrical factor of this expression $\frac{k^2 k_{,}^{\,2} k_{,,}^{\,2}}{a^2 b^2 c^2}$ will appear in the coefficients of y^2 and z^2, and may therefore be eliminated by division from the equation of the cone. Hence the coefficient of x^2 may thus be reduced to

$$\frac{p^2 (k^2 - k_{,}^{\,2})(k^2 - k_{,,}^{\,2})}{k^2 a_{,}^{\,2} b_{,}^{\,2} c_{,}^{\,2}} \quad \ldots\ldots\ldots\ldots\ldots (22).$$

Let A, B be the semiaxes of the section of the ellipsoid $(a_{,} b_{,} c_{,})$ conjugate to the diameter passing through the vertex of the cone, and we shall have, by a well-known relation,

$$p.A.B = a_{,} b_{,} c_{,} \ldots\ldots\ldots\ldots\ldots\ldots (23).$$

Hence (22) may now be reduced to

$$\frac{(k^2 - k_{,}^{\,2})(k^2 - k_{,,}^{\,2})}{k^2 A^2 B^2} \quad \ldots\ldots\ldots\ldots (24).$$

From (15) it follows that

$$\alpha^2 + \beta^2 + \gamma^2 = a^2 + b^2 + c^2 + k^2 + k_{,}^{\,2} + k_{,,}^{\,2} \ldots\ldots (25),$$

and by a common theorem

$$\alpha^2 + \beta^2 + \gamma^2 + A^2 + B^2 = a_{,}^{\,2} + b_{,}^{\,2} + c_{,}^{\,2} = a^2 + b^2 + c^2 + 3k^2 \ldots (26).$$

Hence, combining (25) and (26),

$$A^2 + B^2 = (k^2 - k_{,}^{\,2}) + (k^2 - k_{,,}^{\,2}) \ldots\ldots\ldots\ldots (27).$$

Now the confocal surfaces $(a_{,} b_{,} c_{,})$ and $(a_{,,} b_{,,} c_{,,})$ intersect in a line of curvature, and for the whole of this line A is constant, or when k and $k_{,}$ are constant A is constant; hence

$$A^2 = (k^2 - k_{,}^{\,2}), \quad B^2 = (k^2 - k_{,,}^{\,2}) \ldots\ldots\ldots\ldots (28),$$

and the coefficient of x^2 in the equation of the cone, namely

$$\frac{(k^2 - k_{,}^{\,2})(k^2 - k_{,,}^{\,2})}{k^2 A^2 B^2},$$

becomes simply

$$\frac{1}{k^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (29).$$

By a similar process we shall find the coefficients of y^2 and z^2 to be $\frac{1}{k_i^2}$ and $\frac{1}{k_{ii}^2}$. Hence we derive the following very simple equation for the enveloping cone,

$$\frac{x^2}{k^2} + \frac{y^2}{k_i^2} + \frac{z^2}{k_{ii}^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (30).$$

If we turn to (15) we shall easily perceive that *one* of the factors in the numerator of the value of β^2 must be negative. Let k_i^2 therefore be taken with a negative sign, and let it be greater than b^2. In order that the value of γ^2 may be real, since one of the factors of the numerator is negative, *two* must be negative, hence k_{ii}^2 must be taken with a negative sign, and that there may *not* be *two* negative factors in the value of β^2, k_{ii}^2 must be less than b^2. Now if $a > b > c$ in the order of magnitude, we shall have

$$b_{ii}^2 = b^2 - k_i^2, \quad c_{ii}^2 = c^2 - k_i^2.$$

Hence b_{ii}^2 and c_{ii}^2 must be taken with negative signs. Since $b_{iii}^2 = b^2 - k_{ii}^2$, $c_{iii}^2 = c^2 - k_{ii}^2$, b_{iii}^2 must be taken with a positive sign and c_{iii} with a negative sign. Therefore the surface $(a_i b_i c_i)$ is an ellipsoid, the surface $(a_{ii} b_{ii} c_{ii})$ is an hyperboloid of two sheets, and $(a_{iii} b_{iii} c_{iii})$ is an hyperboloid of one sheet. (30) may now be written

$$\frac{y^2}{k_i^2} + \frac{z^2}{k_{ii}^2} - \frac{x^2}{k^2} = 0.$$

It is remarkable that the constants in the equation of this cone are independent of *abc*, the semiaxes of the surface enveloped. Hence, so long as the enveloped surface remains confocal to the three others, it may change in any manner the ratio or magnitude of its axes without changing the species of the enveloping cone.

When $c = 0$, we get the equation of the cone which stands on the ellipse (a, b) as base, and whose vertex is at the point Q.

II. If now with reference to the new system of coordinates as axes, and with the point Q as centre, we conceive a second group of three surfaces confocal to each other, and a fourth confocal surface whose semiaxes shall be k, k_i, k_{ii}, the squares of the semiaxes of the intersecting group being

$$\left.\begin{array}{lll} k^2 + a^2, & k_i^2 + a^2, & k_{ii}^2 + a^2 \\ k^2 + b^2, & k_i^2 + b^2, & k_{ii}^2 + b^2 \\ k^2 + c^2, & k_i^2 + c^2, & k_{ii}^2 + c^2 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots (31)$$

respectively, these surfaces will intersect in the origin O of the first system of coordinates, and the fourth confocal surface (kk_ik_{ii}) may be enveloped by a cone whose vertex is at O and whose equation referred to the original axes of coordinates is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (32).$$

The original axes of coordinates Ox, Oy, Oz are normals to the second group of confocal surfaces, as Qx', Qy', Qz' are normals to the first, and the sums of the squares of the nine semiaxes in each group will obviously be equal to each other, as also an axis in each pair of corresponding surfaces. It is also obvious, from an inspection of (15), that α, β, γ, the coordinates of the point Q in the first system, become the perpendiculars from the point Q, the origin of the second system, on the tangent planes to the second group of surfaces having their common point of intersection at O.

III. *Let two cones having their common vertex on a surface of the second order, an ellipsoid suppose, $(a_ib_ic_i)$ envelop two confocal surfaces. The diametral plane of the surface conjugate to the diameter passing through the common vertex of the two cones will cut off from their common side a constant length, independent of the position of the common vertex of the two cones on the surface $(a_ib_ic_i)$.*

Let a, b, c; α, β, γ be the semiaxes of the confocal surfaces. And, as in the preceding theorem, let $(a_ib_ic_i)$, $(a_{ii}b_{ii}c_{ii})$, $(a_{iii}b_{iii}c_{iii})$ be the axes of three confocal surfaces passing through the vertex of the cones. Hence we shall have, as in (13),

$$\left.\begin{array}{lll} a_i^2=a^2+k^2, & a_{ii}^2=a^2+k_i^2, & a_{iii}^2=a^2+k_{ii}^2 \\ a_i^2=\alpha^2+h^2, & a_{ii}^2=\alpha^2+h_i^2, & a_{iii}^2=\alpha^2+h_{ii}^2 \end{array}\right\} \ldots(33).$$

The equations of the cones will be, as in (30),

$$\frac{x^2}{k^2}+\frac{y^2}{k_i^2}+\frac{z^2}{k_{ii}^2}=0\,; \quad \frac{x^2}{h^2}+\frac{y^2}{h_i^2}+\frac{z^2}{h_{ii}^2}=0\,;$$

or as these equations may manifestly be written,

$$\left.\begin{array}{l} \dfrac{x^2}{a_i^2-a^2}+\dfrac{y^2}{a_{ii}^2-a^2}+\dfrac{z^2}{a_{iii}^2-a^2}=0 \\ \dfrac{x^2}{a_i^2-\alpha^2}+\dfrac{y^2}{a_{ii}^2-\alpha^2}+\dfrac{z^2}{a_{iii}^2-\alpha^2}=0 \end{array}\right\} \ldots\ldots\ldots (34).$$

Now the distance between the vertex and the diametral plane is p, and as p coincides with the new axis of x, we shall have

$$x = p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (35).$$

Let D be the length of the common side of the cones, then

$$D^2 = x^2 + y^2 + z^2.$$

And if we find the values of x, y, z from the equations (34) and (35), we shall have

$$\frac{D^2}{p^2} = \frac{a_i^2 a_{ii}^2 (a_i^2 - a_{ii}^2) + a_{ii}^2 a_{iii}^2 (a_{ii}^2 - a_{iii}^2) + a_{iii}^2 a_i^2 (a_{iii}^2 - a_i^2)}{(a_{ii}^2 - a_{iii}^2)(a_i^2 - a^2)(a_i^2 - \alpha^2)} \quad \ldots (36).$$

In (23) it was shewn that

$$p^2 = \frac{a_i^2 b_i^2 c_i^2}{(k^2 - k_i^2)(k^2 - k_{ii}^2)};$$

or as $\quad k^2 - k_i^2 = a_i - a_{ii}^2, \quad k^2 - k_{ii}^2 = a_i^2 - a_{iii}^2,$

$$p^2 = \frac{a_i^2 b_i^2 c_i^2}{(a_i^2 - a_{ii}^2)(a_i^2 - a_{iii}^2)}.$$

Now the numerator of (36) may be resolved into the product of the three factors

$-(a_{ii}^2 - a_{iii}^2)(a_{iii}^2 - a_i^2)(a_i^2 - a_{ii}^2)$, and $a_i^2 - a^2 = h^2$, $a_i^2 - \alpha^2 = k^2$.

Hence, making the substitutions indicated,

$$D = \frac{a_i b_i c_i}{kh} \quad \ldots\ldots\ldots\ldots\ldots (37).$$

Hence, as the value of D is independent of k_i, k_{ii} and of h_i, h_{ii}, it will therefore not depend on the two auxiliary confocal surfaces introduced, but the value will continue unchanged wherever the point be taken on the surface of the ellipsoid. Hence, D^2 varies inversely as the product of the squares of the coincident semiaxes, for

$$k^2 = a_i^2 - a^2, \quad h^2 = a_i^2 - \alpha^2.$$

When the enveloped surfaces become plane sections, $c = 0$, $\beta = 0$, but $b_i^2 = \beta^2 + h^2$, $c_i^2 = c^2 + k^2$; hence, in this case $b_i^2 = h^2$, $c_i^2 = k^2$, or $D = a_i$.

IV. *A cone whose vertex is on a surface of the second order envelopes a confocal surface. To determine the length of the axis of the cone between the vertex and the plane of contact.*

Let the equation of the locus of the vertex of the cone be

$$\frac{x_i^2}{a^2} + \frac{y_i^2}{b^2} + \frac{z_i^2}{c^2} = 1 \ldots\ldots\ldots\ldots (38),$$

$(x_1y_1z_1)$ being the vertex of the cone.

Let $$\frac{x^2}{a^2+k^2}+\frac{y^2}{b^2+k^2}+\frac{z^2}{c^2+k^2}=1 \ldots\ldots\ldots(39),$$

be the equation of the confocal surface.

The equation of the polar plane of $(x_1y_1z_1)$ with reference to this last surface is

$$\frac{xx_1}{a^2+k^2}+\frac{yy_1}{b^2+k^2}+\frac{zz_1}{c^2+k^2}=1 \ldots\ldots\ldots(40).$$

The equations to the normal at the point $(x_1y_1z_1)$ in (38) are

$$x-x_1=\frac{\cos\lambda}{\cos\nu}(z-z_1),\quad y-y_1=\frac{\cos\mu}{\cos\nu}(z-z_1)\ldots(41),$$

but $$\frac{\cos\lambda}{\cos\nu}=\frac{c^2x_1}{a^2z_1},\quad \frac{\cos\mu}{\cos\nu}=\frac{c^2y_1}{b^2z_1}.$$

Now $$\Delta^2=(x-x_1)^2+(y-y_1)^2+(z-z_1)^2,$$

or substituting in this expression the values derived from the preceding equations,

$$\Delta^2=\left[\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right]\frac{c^4(z-z_1)^2}{z_1^2} \ldots\ldots(42).$$

We must now determine the value of z for the point in which the axis of the cone meets the polar plane. For this purpose, from the equation of the polar plane

$$\frac{xx_1}{a^2+k^2}+\frac{yy_1}{b^2+k^2}+\frac{zz_1}{c^2+k^2}=1,$$

subtract the identity

$$\frac{x_1^2}{a^2+k^2}+\frac{y_1^2}{b^2+k^2}+\frac{z_1^2}{c^2+k^2}=\frac{x_1^2}{a^2+k^2}+\frac{y_1^2}{b^2+k^2}+\frac{z_1^2}{c^2+k^2};$$

replacing 1 by its value

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2},$$

the result will be found

$$\frac{x_1(x-x_1)}{a^2+k^2}+\frac{y_1(y-y_1)}{b^2+k^2}+\frac{z_1(z-z_1)}{c^2+k^2}$$
$$=k^2\left[\frac{x_1^2}{a^2(a^2+k^2)}+\frac{y_1^2}{b^2(b^2+k^2)}+\frac{z_1^2}{c^2(c^2+k^2)}\right];$$

or putting for $(x - x_{,})$ and $(y - y_{,})$ their values derived from the equations of the normal (41), we find

$$c^2\left(\frac{z-z_{,}}{z_{,}}\right) = k^2.$$

Whence combining this expression with (42), we find

$$\Delta^2 = k^4\left[\frac{x_{,}^2}{a^4} + \frac{y_{,}^2}{b^4} + \frac{z_{,}^2}{c^4}\right],$$

or as this latter expression is equal to $\frac{1}{p^2}$, finally

$$\Delta = \frac{k^2}{p} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43);$$

for any other confocal surface, the vertex of the cone remaining unchanged,

$$\Delta' = \frac{k_{,}^2}{p},$$

or $$\Delta : \Delta' :: k^2 : k_{,}^2.$$

V. *To transform the equation of a surface of the second order, so that the axes of coordinates shall be the normal to the surface at a given point, and the two right lines in the tangent plane at this point, which are tangents to the lines of greatest and least curvature.*

Let the normal be the axis of x, then the axes of coordinates are the normals to three confocal surfaces passing through this point. Now if α, β, γ are the coordinates of this new origin on the surface, substituting the values of x, y, z in the equation of the surface

$$\frac{x^2}{a^2 + k^2} + \frac{y^2}{b^2 + k^2} + \frac{z^2}{c^2 + k^2} = 1,$$

derived from (11), we shall find the following resulting expression,

$$\left.\begin{aligned}
&\frac{\alpha^2}{a_{,}^2} + \frac{2\alpha^2}{a_{,}^2}\left[\frac{px_{,}}{a_{,}^2} + \frac{py_{,}}{a_{,,}^2} + \frac{pz_{,}}{a_{,,,}^2}\right] + \frac{\alpha^2}{a_{,}^2}\left[\frac{px_{,}}{a_{,}^2} + \frac{py_{,}}{a_{,,}^2} + \frac{pz_{,}}{a_{,,,}^2}\right]^2 \\
&\frac{\beta^2}{b_{,}^2} + \frac{2\beta^2}{b_{,}^2}\left[\frac{px_{,}}{b_{,}^2} + \frac{py_{,}}{b_{,,}^2} + \frac{pz_{,}}{b_{,,,}^2}\right] + \frac{\beta^2}{b_{,}^2}\left[\frac{px_{,}}{b_{,}^2} + \frac{py_{,}}{b_{,,}^2} + \frac{pz_{,}}{b_{,,,}^2}\right]^2 \\
&\frac{\gamma^2}{c_{,}^2} + \frac{2\gamma^2}{c_{,}^2}\left[\frac{px_{,}}{c_{,}^2} + \frac{py_{,}}{c_{,,}^2} + \frac{pz_{,}}{c_{,,,}^2}\right] + \frac{\gamma^2}{c_{,}^2}\left[\frac{px_{,}}{c_{,}^2} + \frac{py_{,}}{c_{,,}^2} + \frac{pz_{,}}{c_{,,,}^2}\right]^2
\end{aligned}\right\} = 1 \ldots (44).$$

Adding these terms vertically, the sum of the first column is manifestly = 1. The sum of the terms in the second column is $\frac{2x_{,}}{p}$. The sum of the terms in the third column is

$$p_{,}y_{,}\left\{\frac{\alpha^2}{a_{,}^2a_{,,}^2}+\frac{\beta^2}{b_{,}^2b_{,,}^2}+\frac{\gamma^2}{c_{,}^2c_{,,}^2}\right\}.$$

Now the cosines of the angles λ, μ, ν, which the axes of coordinates make with the perpendicular p (let fall on the tangent plane through the point $(\alpha\beta\gamma)$ on the surface $(a_{,}b_{,}c_{,})$,) are $\frac{p\alpha}{a_{,}^2}$, $\frac{p\beta}{b_{,}^2}$, $\frac{p\gamma}{c_{,}^2}$; and the cosines of the angles λ', μ', ν' which $p_{,}$ makes, the perpendicular on the tangent plane through the point $(\alpha\beta\gamma)$ on the surface $(a_{,,}b_{,,}c_{,,})$ are $\frac{p_{,}\alpha}{a_{,,}^2}$, $\frac{p_{,}\beta}{b_{,,}^2}$, $\frac{p_{,}\gamma}{c_{,,}^2}$; and as these planes are at right angles,

$$\cos\lambda\cos\lambda'+\cos\mu\cos\mu'+\cos\nu\cos\nu'=0.$$

Hence the third column in the coefficient of $y_{,}=0$. In like manner the fourth column on the coefficient of $z_{,}=0$. The fifth column is

$$p^2x_{,}^2\left\{\frac{\alpha^2}{a_{,}^6}+\frac{\beta^2}{b_{,}^6}+\frac{\gamma^2}{c_{,}^6}\right\}.$$

Now as

$$\cos^2\lambda=\frac{p^2\alpha^2}{a_{,}^4},\quad \cos^2\mu=\frac{p^2\beta^2}{b_{,}^4},\quad \cos^2\gamma=\frac{p^2\gamma^2}{c_{,}^4},$$

the coefficient of $x_{,}^2$ may be written

$$\frac{\cos^2\lambda}{a_{,}^2}+\frac{\cos^2\mu}{b_{,}^2}+\frac{\cos^2\nu}{c_{,}^2}.$$

This expression is $=\frac{1}{r^2}$, if we denote by r the semi-diameter of the surface parallel to p.

In like manner the coefficients of $y_{,}^2$ and $z_{,}^2$ are $\frac{1}{r_{,}}$ and $\frac{1}{r_{,,}}$ respectively; $r_{,}$ and $r_{,,}$ being parallel to $p_{,}$ and $p_{,,}$.

The coefficient of $x_{,}y_{,}$ is

$$2pp_{,}\left[\frac{\alpha^2}{a_{,}^4a_{,,}^2}+\frac{\beta^2}{b_{,}^4b_{,,}^2}+\frac{\gamma^2}{c_{,}^4c_{,,}^2}\right];$$

multiply the terms of this expression by the equivalent factors $a_{,,}^2-a_{,}^2=b_{,}^2-b_{,}^2=c_{,,}^2-c_{,}^2=k_{,,}^2-k^2$, dividing by

this latter, and the expression will be transformed into

$$\frac{2pp_{,}}{k_{,}^2 - k^2}\left[\frac{\alpha^2}{a_{,}^4} + \frac{\beta^2}{b_{,}^4} + \frac{\gamma^2}{c_{,}^4} - \left(\frac{\alpha^2}{a_{,}^2 a_{,,}^2} + \frac{\beta^2}{b_{,}^2 b_{,,}^2} + \frac{\gamma^2}{c_{,}^2 c_{,,}^2}\right)\right].$$

Now the first of these groups is equal to $\frac{1}{p^2}$, and the second, as we have already shewn, is $= 0$; hence, the coefficient of xy is $\frac{2p_{,}}{p(k_{,}^2 - k^2)}$.

In the same manner it may be shewn that the coefficient of $x_{,}z_{,}$ is $\frac{2p_{,,}}{p(k_{,,}^2 - k^2)}$.

Let $r_{,}$ and $r_{,,}$ be the axes of the section parallel to the tangent plane at the point $(\alpha\beta\gamma)$, then, as we have found in (28),

$$r_{,,}^2 = k^2 - k_{,,}^2, \quad r_{,}^2 = k^2 - k_{,}^2.$$

Introducing into the equation (44) the resulting expressions thus found, the equation of the surface will at length become

$$\frac{x^2}{r^2} + \frac{y^2}{r_{,}^2} + \frac{z^2}{r_{,,}^2} - \frac{2p_{,}}{pr_{,}^2}xy - \frac{2p_{,,}}{pr_{,,}^2}xz + \frac{2x}{p} = 0 \ldots (45).$$

In this equation the coefficients are the perpendiculars p, $p_{,}$, $p_{,,}$ from the centre on the coordinate planes, and the three diameters of the surface which coincide with these perpendiculars.

Let $x = 0$, then the equation becomes

$$y = \sqrt{(-1)}\, z\, \frac{r_{,}}{r_{,,}},$$

which can be real only when one of the semiaxes $r_{,}$ or $r_{,,}$ is imaginary, or in other words, when the surface in a continuous hyperboloid. Since

$$y = \sqrt{(-1)}\, z\, \frac{r_{,}}{r_{,,}}, \quad y = z\sqrt{\left(\frac{k_{,}^2 - k^2}{k^2 - k_{,,}^2}\right)} \ldots\ldots$$

These are the generatrices of the hyperboloid, and it may easily be shewn that they are also the focal lines of the cone whose equation is

$$\frac{x^2}{k^2} + \frac{y^2}{k_{,}^2} + \frac{z^2}{k_{,,}^2} = 0.$$

VI. *Along a line of curvature tangent planes are drawn to a surface of the second order. The perpendiculars from the centre on these planes generate a cone of the second order,*

whose focal lines coincide with the optic axes of the surface or with the perpendiculars to its circular sections.

Since a line of curvature is generated by the intersection of two confocal surfaces, let the equations of these surfaces be

$$\frac{x_{,}^2}{a^2}+\frac{y_{,}^2}{b^2}+\frac{z_{,}^2}{c^2}=1, \quad \text{and} \quad \frac{x_{,}^2}{a^2+k^2}+\frac{y_{,}^2}{b^2+k^2}+\frac{z_{,}^2}{c^2+k^2}=1 \ldots (46),$$

Let λ, μ, ν be the angles which p makes with the axes, then

$$\cos\lambda=\frac{px_{,}}{a^2}, \quad \cos\mu=\frac{py_{,}}{b^2}, \quad \cos\nu=\frac{pz_{,}}{c^2};$$

from these five equations eliminating $x,y,z,$ and p, and putting for $\frac{\cos\lambda}{\cos\nu}$, $\frac{\cos\mu}{\cos\nu}$, their values $\frac{x}{z}$, $\frac{y}{z}$, the resulting equation will become

$$\frac{a^2x^2}{a^2+k^2}+\frac{b^2y^2}{b^2+k^2}+\frac{c^2z^2}{c^2+k^2}=0 \ldots\ldots\ldots\ldots (47).$$

Now the angles which the focals of this cone make with the internal axe are given by the equation

$$\tan^2\varepsilon=\frac{c^2(a^2-b^2)}{b^2(a^2-c^2)},$$

a result independent of k; hence, all the cones so generated are confocal. Hence, the supplemental cones to these have their circular sections parallel to those of the given surface, and therefore these supplemental cones are the intersections of spheres with the given surface.

The angles which the optic axes make with the vertical axes are given by the same equation

$$\tan^2\varepsilon_{,}=\frac{c^2(a^2-b^2)}{b^2(a^2-c^2)}, \quad \text{or } \varepsilon_{,}=\varepsilon.$$

NOTES ON MOLECULAR MECHANICS.

By the Rev. Samuel Haughton.

No. III. Normal and Transverse Vibrations.

In my last Note, Vol. viii. p. 159, I have given the general equations of motion of an elastic system, derived from the function of the second order of the differential coefficients of the displacements; together with the integral

corresponding to plane waves. I shall now restrict the results there obtained, so that the vibrations shall be normal and transversal.

If, in equations (10), we substitute (l, m, n) for $(\cos\alpha, \cos\beta, \cos\gamma)$, and eliminate v^2, we shall have the conditions necessary to express that one of the vibrations is normal to the wave plane and the other two transverse; the resulting equations are

$$\left.\begin{aligned}(Q'-R')mn + F''(n^2-m^2) + H'ln - G'lm &= 0\\ (R'-P')nl + G''(l^2-n^2) + F'ml - H'mn &= 0\\ (P'-Q')lm + H''(m^2-l^2) + G'nm - F'nl &= 0\end{aligned}\right\} \ldots(24).$$

Introducing into these equations the values of P', Q', &c., we find the following twenty-four conditions:

$$\left.\begin{aligned}(a_1b_1) = (a_1c_1) = (b_2c_2) &= 0\\ (a_2b_2) = (a_3c_3) = (b_3c_3) &= 0\\ (b_1c_1) + (a_1c_2) + (a_2c_1) = (b_1c_1) + (a_1b_3) + (a_3b_1) &= 0\\ (c_2a_2) + (b_2a_3) + (b_3a_2) = (c_2a_2) + (b_2c_1) + (b_1c_2) &= 0\\ (a_3b_3) + (c_3b_1) + (c_1b_3) = (a_3b_3) + (c_3a_2) + (c_2a_3) &= 0\\ (a_1a_2) = (b_1b_2) = (c_1c_2) - (a_3b_3) &\\ (b_2b_3) = (c_2c_3) = (a_2a_3) - (b_1c_1) &\\ (c_3c_1) = (a_3a_1) = (b_3b_1) - (a_2c_2) &\\ (a_1^2) = (b_1^2) + (a_1b_3) + (a_2b_1) = (c_1^2) + (a_1c_3) + (a_3c_1) &\\ (b_2^2) = (c_2^2) + (b_2c_3) + (b_2c_2) = (a_2^2) + (b_2a_1) + (b_1a_2) &\\ (c_3^2) = (a_3^2) + (c_3a_1) + (c_1a_3) = (b_3^2) + (c_3b_2) + (c_2b_3) &\end{aligned}\right\} \ldots(25).$$

The general function V of the second order is

$$2V = \Sigma(\alpha_1^2) + 2\Sigma(\alpha_1\alpha_2) + 2\Sigma(\alpha_1\beta_1) + 2\Sigma(\alpha_1\beta_2) \ldots(26),$$

and contains forty-five coefficients, which are reduced by equations (25) to twenty-one coefficients.

It has been proved by Mr. Jellett that the coefficients of the last term of (26), viz. $\Sigma(\alpha_1\beta_2)$, will be equal in pairs, or reduced from eighteen to nine on the following supposition:

"*If, in a system of molecules, the forces developed by the displacement of two molecules be a function of their relative displacements only, and tend to restore them to their original positions, the function V for such a system will con-*

tain only thirty-six coefficients, the coefficients of $\Sigma(\alpha_1\beta_2)$ *being equal in pairs.*"

As such a supposition appears to be, *a priori*, very probable, I shall adopt it in the investigations which follow in this note, and consider the consequences which follow from it.

It supplies us with the following nine equations:

$$\left.\begin{aligned} (b_2c_3) = (b_3c_2), \quad c_2a_3 = c_3a_2, \quad a_2b_3 = a_3b_2 \\ (b_3c_1) = (b_1c_3), \quad c_3a_1 = c_1a_3, \quad a_3b_1 = a_1b_3 \\ (b_1c_2) = (b_2c_1), \quad c_1a_2 = c_2a_1, \quad a_1b_2 = a_2b_1 \end{aligned}\right\} \ldots (27).$$

Combining these conditions with (25), and introducing them into (26), I find

$$\left.\begin{aligned} 2V = A\lambda + B\mu + C\nu + 2D\phi + 2E\chi + 2F\psi \\ + P(X^2 - u_1) + Q(Y^2 - v_2) + R(Z^2 - w_3) \\ + L\{2YZ - (v_3 + w_2)\} + M\{2XZ - (w_1 + u_3)\} \\ + N\{2XY - (u_2 + v_1)\} \end{aligned}\right\} \ldots (28),$$

where

$$\left.\begin{aligned} A = (a_1^2), \quad D = (a_1a_2), \\ B = (b_2)^2, \quad E = (b_2b_3), \\ C = (c_3)^2, \quad F = (c_1c_3), \\ P = -2(b_3c_3), \quad L = -(b_1c_1), \\ Q = -2(c_3a_1), \quad M = -(c_2a_2), \\ R = -2(a_1b_2), \quad N = -(a_3b_3), \\ \lambda = \alpha_1^2 + \beta_1^2 + \gamma_1^2, \quad \phi = \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2, \\ \mu = \alpha_2^2 + \beta_2^2 + \gamma_2^2, \quad \chi = \alpha_3\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3, \\ \nu = \alpha_3^2 + \beta_3^2 + \gamma_3^2, \quad \psi = \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3, \\ X = \beta_3 - \gamma_2, \\ Y = \gamma_1 - \alpha_3, \\ Z = \alpha_2 - \beta_1, \\ u_1 = \beta_2\gamma_3 - \beta_3\gamma_2, \quad v_1 = \gamma_2\alpha_3 - \gamma_3\alpha_2, \quad w_1 = \alpha_2\beta_3 - \alpha_3\beta_2 \\ u_2 = \beta_3\gamma_1 - \beta_1\gamma_3, \quad v_2 = \gamma_3\alpha_1 - \gamma_1\alpha_3, \quad w_2 = \alpha_3\beta_1 - \alpha_1\beta_3 \\ u_3 = \beta_1\gamma_2 - \beta_2\gamma_1, \quad v_3 = \gamma_1\alpha_2 - \gamma_2\alpha_1, \quad w_3 = \alpha_1\beta_2 - \alpha_2\beta_1 \end{aligned}\right\} \ldots (29).$$

It is not difficult to prove that if the coordinate axes be changed in direction, the origin remaining the same, that the functions $\lambda, \mu, \nu, \phi, \chi, \psi$ are transformed by the same equations as $x^2, y^2, z^2, yz, xz, xy$; and that $u_1, v_2, w_3, v_3 + w_2, w_1 + u_3, u_2 + v_1$, are transformed by the same equations as

x^2, y^2, z^2, $2yz$, $2xz$, $2xy$; and it is well known that X, Y, Z, may be transformed by the same equations as x, y, z.

Hence the function V may be reduced to a function of nine coefficients in either of two ways, 1st by assuming for coordinate axes the axes of the ellipsoid

$$Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy = 1;$$

or, 2nd, of the ellipsoid

$$Px^2 + Qy^2 + Rz^2 + 2Lyz + 2Mxz + 2Nxy = 1.$$

The position of the first set of axes of elasticity depends on the external conditions to which the body is subject, and the position of the second set of axes appears to depend on the inherent structure of the body.

If we suppose the axes chosen to be the axes of the second ellipsoid, so as to destroy the coefficients L, M, N, and seek the equation of the surface of *Wave Slowness* (12), we shall find, after some reductions,

$$(S-1)\left\{\begin{array}{l}(x^2+y^2+z^2)\,(QRx^2+PRy^2+PQz^2)\\ \quad+(S-1)\,\{(Q+R)\,x^2+(P+R)\,y^2+(P+Q)\,z^2\}\\ \quad+(S-1)^2\end{array}\right\}=0\ldots(30),$$

where

$$S = Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy.$$

Equation (30) is composed of two factors; one of the second degree, to which the normal vibration corresponds, and the other of the fourth degree, to the two sheets of which the transverse vibrations belong.

I am informed by Mr. Salmon, that this surface of the fourth degree has sixteen multiple points, and consequently that its reciprocal, or the wave surface, will be reduced from the thirty-sixth to the fourth degree. If we suppose $S = 0$, the surface of the fourth degree becomes

$$\left.\begin{array}{l}(x^2+y^2+z^2)\,(QRx^2+PRy^2+PQz^2)\\ \quad-\{(Q+R)\,x^2+(P+R)\,y^2+(P+Q)\,z^2\}+1\end{array}\right\}=0\ldots(31),$$

which is the well-known equation of Fresnel's Wave Surface.

I shall now proceed to shew that the supposition $S = 0$ is not an allowable one, and that consequently the wave surface derived from the function (28) can never become accurately Fresnel's Wave Surface.

Let us suppose the body to be homogeneous and uncrystalline; which will reduce (28) to the following form,

$$-2V = A(\lambda+\mu+\nu) + P(X^2+Y^2+Z^2-u_1-v_2-w_3)\ldots(32).$$

Hence we find

$$\left.\begin{aligned}
-\frac{dV}{d\alpha_1} &= A\alpha_1 - \tfrac{1}{2}P(\beta_2+\gamma_3)\\
-\frac{dV}{d\beta_2} &= A\beta_2 - \tfrac{1}{2}P(\gamma_3+\alpha_1)\\
-\frac{dV}{d\gamma_3} &= A\gamma_3 - \tfrac{1}{2}P(\alpha_1+\beta_2)\\
-\frac{dV}{d\alpha_2} &= (A+P)\,\alpha_2 - \tfrac{1}{2}P\beta_1\\
-\frac{dV}{d\beta_1} &= (A+P)\,\beta_1 - \tfrac{1}{2}P\alpha_2\\
-\frac{dV}{d\beta_3} &= (A+P)\,\beta_3 - \tfrac{1}{2}P\gamma_2\\
-\frac{dV}{d\gamma_2} &= (A+P)\,\gamma_2 - \tfrac{1}{2}P\beta_3\\
-\frac{dV}{d\gamma_1} &= (A+P)\,\gamma_1 - \tfrac{1}{2}P\alpha_3\\
-\frac{dV}{d\alpha_3} &= (A+P)\,\alpha_3 - \tfrac{1}{2}P\gamma_1
\end{aligned}\right\} \quad \ldots\ldots\ldots (33).$$

As these coefficients are the forces acting on the faces of the elementary parallelepiped (vol. v. p. 176), it is necessary that they should have opposite signs to those of the displacements and strains, by which they are produced, and consequently the right-hand members of these equations must be essentially positive.

Hence, adding the first three together, we find

$$A - P > 0,$$

and adding the last six together in pairs, we find

$$A + \tfrac{1}{2}P > 0.$$

These equations, taken together, prove that A must be positive and cannot be zero or negative; and that P may have any value lying between $+A$ and $-2A$, which limits, positive and negative, it cannot exceed. From this investigation it appears that A cannot vanish without P vanishing at the same time in an uncrystalline body, and similar reasoning would apply to the case of a crystallized body.

The equations of motion deducible from (32) may be written thus:

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= T^2\left(\frac{d^2\xi}{dx^2}+\frac{d^2\xi}{dy^2}+\frac{d^2\xi}{dz^2}\right)+(N^2-T^2)\frac{d\omega}{dx} \\ \frac{d^2\eta}{dt^2} &= \text{.....} \\ \frac{d^2\zeta}{dt^2} &= \text{.....} \end{aligned}\right\} \text{...(34),}$$

where $\omega = \frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}$, $\frac{A}{\varepsilon} = N^2$, and $\frac{P}{\varepsilon} = T^2 - N^2$.

Equations (34) may be transformed into the following:

$$\left.\begin{aligned} \frac{d^2\omega}{dt^2} &= N^2\left(\frac{d^2\omega}{dx^2}+\frac{d^2\omega}{dy^2}+\frac{d^2\omega}{dz^2}\right) \\ \frac{d^2X}{dt^2} &= T^2\left(\frac{d^2X}{dx^2}+\frac{d^2X}{dy^2}+\frac{d^2X}{dz^2}\right) \\ \frac{d^2Y}{dt^2} &= T^2\left(\frac{d^2Y}{dx^2}+\frac{d^2Y}{dy^2}+\frac{d^2Y}{dz^2}\right) \\ \frac{d^2Z}{dt^2} &= T^2\left(\frac{d^2Z}{dx^2}+\frac{d^2Z}{dy^2}+\frac{d^2Z}{dz^2}\right) \end{aligned}\right\} \text{..... (35).}$$

These are the equations of wave propagation, which are usually adopted and have been derived by different writers from different principles.

N and T in these equations denote the normal and transverse wave velocities respectively; and from the conditions of stability already found, we obtain

$$\left.\begin{aligned} 2N^2 &> T^2 \\ N^2 + T^2 &> 0 \end{aligned}\right\} \text{.....(36),}$$

from which we may infer that in a body such as is here considered, it is not necessary that the normal wave velocity should be greater than the transverse, and that the latter may exceed the former in the proportion of $\sqrt{2}$ to 1.

The existence of a normal vibration has been denied by many writers on Physical Optics, on the ground that, if it existed, it would produce double refraction in a prism made of glass or any other uncrystalline substance. This objection has been answered in two ways; first, by assuming the velocity of the normal vibration to be equal to that of the

transverse vibration; and secondly, by supposing the normal vibration to be evanescent. I have just shewn, that in a body constituted as I have supposed, it is possible for the normal velocity to become equal to the transverse velocity, without endangering the stability of the system; and I shall now inquire whether the normal velocity may be considered equal to the transverse velocity in the problem of refraction.

Let the axis of z be perpendicular to the surface, the plane (x, z) being the plane of incidence, and let the incident vibration be confined to the plane of incidence. By (34), since $N = T$, we have

$$\left\{\frac{d^2}{dt^2} - T^2\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right)\right\}\{\xi, \eta, \zeta\} = 0$$

for the equations of wave propagation, and by comparing (33) and (5), (6), we find for the conditions at the limits

$$\xi_0 = \xi_0', \quad \zeta_0 = \zeta_0', \quad A\gamma_3 = A'\gamma_3', \quad A\alpha_3 = A'\alpha_3' \ldots (37).$$

Let τ, τ_1, τ_{11} denote the incident, reflected transverse, and reflected normal vibration; and τ', τ'' the refracted transverse and refracted normal vibration; then (37) may be written thus:

$$\left.\begin{aligned}
&\tau\cos\alpha + \tau_1\cos\alpha_1 + \tau_{11}\cos\alpha_{11} = \tau'\cos\alpha' + \tau''\cos\alpha'' \\
&\tau\cos\gamma + \tau_1\cos\gamma_1 + \tau_{11}\cos\gamma_{11} = \tau'\cos\gamma' + \tau''\cos\gamma'' \\
&A\left(\frac{\tau}{\lambda}n\cos\gamma + \frac{\tau_1}{\lambda_1}n_1\cos\gamma_1 + \frac{\tau_{11}}{\lambda_{11}}n_{11}\cos\gamma_{11}\right) \\
&\qquad = A'\left(\frac{\tau'}{\lambda'}n'\cos\gamma' + \frac{\tau''}{\lambda''}n''\cos\gamma''\right) \\
&A\left(\frac{\tau}{\lambda}n\cos\alpha + \frac{\tau_1}{\lambda_1}n_1\cos\alpha_1 + \frac{\tau_{11}}{\lambda_{11}}n_{11}\cos\alpha_{11}\right) \\
&\qquad = A'\left(\frac{\tau'}{\lambda'}n'\cos\alpha' + \frac{\tau''}{\lambda''}n''\cos\alpha''\right)
\end{aligned}\right\} \ldots (38),$$

and the angles contained in these equations are connected with the angles of incidence and refraction of the respective rays in the following way:

$$\left.\begin{aligned}
&\cos\alpha = \cos i, \quad n = -\cos i, \quad l_{11} = \cos\alpha_{11} = \sin i \\
&\cos\gamma = \sin i, \quad l = \sin i, \quad n_{11} = \cos\gamma_{11} = \cos i \\
&\cos\alpha_1 = \cos i, \quad n_1 = \cos i, \\
&\cos\gamma_1 = -\sin i, \quad l_1 = \sin i, \quad l'' = \cos\alpha'' = \sin r \\
&\cos\alpha' = \cos r, \quad n' = -\cos r, \quad n'' = \cos\gamma'' = -\cos r \\
&\cos\gamma' = \sin r, \quad l' = \sin r
\end{aligned}\right\} \ldots (39),$$

Introducing these values into equations (38), and remembering that the normal and transverse velocities are equal, we find

$$\left.\begin{aligned}
&(\tau+\tau_{,})\cos i+\tau_{,,}\sin i=\tau'\cos r+\tau''\sin r\\
&(\tau-\tau_{,})\sin i+\tau_{,,}\cos i=\tau'\sin r-\tau''\cos r\\
&(\tau+\tau_{,})\sin i\cos i-\tau_{,,}\cos^2 i\\
&\qquad=k\mu\{\tau'\sin r\cos r-\tau''\cos^2 r\}\\
&(\tau-\tau_{,})\cos^2 i-\tau_{,,}\sin i\cos i\\
&\qquad=k\mu\{\tau'\cos^2 r+\tau''\sin r\cos r\}
\end{aligned}\right\}\ldots\ldots(40),$$

where $k=\dfrac{A'}{A}$, and μ is the refractive index $=\dfrac{\sin i}{\sin r}$.

If, to simplify the calculations, we make $k=1$,* equations (40) may be reduced to the following:

$$(\tau+\tau_{,})\sin 2i-\tau_{,,}\cos 2i=2\tau'\sin i\cos r-\mu\tau''\cos 2r,$$
$$(\tau-\tau_{,})\cos 2i-\tau_{,,}\sin 2i=\tau'\mu\cos 2r+2\tau''\sin i\cos r,$$
$$\tau-\tau_{,}=\mu\tau',$$
$$\tau_{,,}=\mu\tau'',$$

which may be converted into the following:

$$(\tau+\tau_{,})\sin 2i-(\tau-\tau_{,})\sin 2r=\mu\tau''(\cos 2i-\cos 2r),$$
$$(\tau-,\tau)(\cos 2i-\cos 2r)\qquad=\mu\tau''(\sin 2i+\sin 2r),$$

from which may be derived the following:

$$\frac{\tau_{,}}{\tau}=\frac{Q-\tan(i-r)}{Q+\tan(i+r)}\ldots\ldots\ldots\ldots\ldots\ldots(41),$$

where $$Q=\tan(i+r)\tan^2(i-r).$$

Equation (41) represents the amplitude of the transverse reflected vibration, the incident transverse vibration being supposed to be unity. When the refractive index is small, and therefore the difference between i and r also small, the quantity Q, which depends on the square of the tangent of this difference, will diminish more rapidly than the

* This assumption is one which has been often made, and supposes the velocity of wave propagation to vary inversely as the square root of the density.

second part of the expression (41), which depends on the first power of the tangent; and therefore when the refractive index is small, the numerical values of (41) will be nearly represented by the formula

$$\frac{\tau_1}{\tau} = -\frac{\tan(i-r)}{\tan(i+r)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (42),$$

which coincides with the well-known law of Brewster and Fresnel.

If the incident vibration be perpendicular to the plane of incidence, the conditions at the limits will be

$$\eta_0 = \eta_0', \quad (A\beta_3)_0 = (A'\beta'_3)_0 \quad \ldots\ldots\ldots\ldots (43).$$

These equations give, assuming as before $A = A'$,

$$\left.\begin{aligned} \tau + \tau_1 &= \tau' \\ (\tau - \tau_1)\cos i &= \mu\tau' \cos r \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (44).$$

From which it is easy to find

$$2\tau \cos i = \tau'(\cos i + \mu \cos r),$$

$$2\tau_1 \cos i = \tau'(\cos i - \mu \cos r).$$

And finally

$$\frac{\tau_1}{\tau} = -\frac{\sin(i-r)}{\sin(i+r)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (45),$$

which is the value given by Brewster and Fresnel's law.

Trinity College, Dublin,
Nov. 12, 1853.

SUR UNE PROPRIÉTÉ D'UN PRODUIT DE FACTEURS LINÉAIRES.

Par Professeur François Brioschi.

Un mémoire de M. Cayley* publié le dernier cahier de ce *Journal* m' à suggéré une remarquable propriété pour le produit d'un nombre déterminé de facteurs linéaires.

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* On the Rationalization of certain Algebraical Equations, May 1853.

En considérant deux éléments x, y, on voit tout de suite, que les seuls résultats essentiellement différents qu'on obtient en les ajoutant l'un à l'autre, sont donnés par les expressions

$$x + y, \quad x - y.$$

Or si l'on suppose

$$x + y = 0, \quad x^2 = a, \quad y^2 = b,$$

et on multiplie les termes de la première par 1 et par xy ; on obtient en observant les deux autres équations

$$x + y = 0,$$

$$bx + ay = 0,$$

et par conséquent

$$\Delta = \begin{vmatrix} 1 & 1 \\ b & a \end{vmatrix} = 0.$$

Si l'on considère l'équation

$$x - y = 0,$$

et on fait sur elle des opérations analogues aux supérieures on a

$$\nabla = \begin{vmatrix} 1, & -1 \\ -b, & a \end{vmatrix} = \Delta = 0.$$

J'observe que l'équation $\Delta = 0$ a son origine dans la $x+y=0$, par conséquent Δ sera divisible par $x+y$: et analoguement ∇ sera divisible par $x-y$, et l'on aura

$$(x + y)(x - y) = \Delta.$$

Si l'on considère trois éléments x, y, z, on aura quatre expressions différentes

$$x + y + z, \quad x + y - z, \quad x - y + z, \quad -x + y + z,$$

et en supposant

$$x + y + z = 0, \quad x + y - z = 0, \quad x - y + z = 0, \quad -x + y + z = 0,$$

$$x^2 = a, \quad y^2 = b, \quad z^2 = c,$$

on obtiendra en multipliant les termes de chacune par $1, yz, xz, zx, xy$,

$$\Delta = \begin{vmatrix} . & 1 & 1 & 1 \\ 1 & . & c & b \\ 1 & c & . & a \\ 1 & b & a & . \end{vmatrix} = 0,$$

et Δ sera divisible par $x+y+z$, $x+y-z$, &c., et on aura

$$(x+y+z)(x+y-z)(x-y+z)(-x+y+z) = -\Delta.$$

Si on représente avec x, y, z les trois côtés d'un triangle, et avec A sa surface on aura

$$\Delta = (4A)^2,$$

et aussi

$$\Delta = \frac{(2A)^4}{r\,r_1 r_2 r_3},$$

r, r_1, r_2, r_3, désignant les rayons des cercles inscrits et ex-inscrits aux triangle. Observons qu'on a évidemment

$$\Delta = \frac{1}{x^2y^2z^2}\begin{vmatrix} . & x & y & z \\ x & . & cxy & bxz \\ y & cxy & . & ayz \\ z & bxz & ayz & . \end{vmatrix} = \begin{vmatrix} . & x & y & z \\ x & . & z & y \\ y & z & . & x \\ z & y & x & . \end{vmatrix}$$

Pour quatre éléments le nombre des expressions différentes étant huit, on aura les équations

$$x+y+z+t=0,\quad -x+y+z+t=0,\quad x-y+z+t=0,\quad x+y-z+t=0,$$
$$x+y+z-t=0,\quad -x+y+z-t=0,\quad -x+y-z+t=0,\quad -x-y+z+t=0,$$

desquelles en posant

$$x^2=a,\quad y^2=b,\quad z^2=c,\quad t^2=d,$$

et en multipliant avec ordre par

$$1,\ zt,\ yt,\ xt,\ zy,\ zx,\ yx,\ xyzt,$$

on aura

$$\Delta = \begin{vmatrix} . & . & . & . & 1 & 1 & 1 & 1 \\ . & . & 1 & 1 & . & . & d & c \\ . & 1 & . & 1 & . & d & . & b \\ . & 1 & 1 & . & d & . & . & a \\ 1 & . & . & 1 & . & c & b & . \\ 1 & . & 1 & . & c & . & a & . \\ 1 & 1 & . & . & b & a & . & . \\ d & c & b & a & . & . & . & . \end{vmatrix} = 0,$$

et Δ sera égal au produit des premiers membres des équations supérieures.

Si x, y, z, t désignent les aires des quatre faces d'un tétrahèdre, V le volume, et $r, r_1, \ldots r_7$ les rayons des sphères

inscrites et ex-inscrites, on aura

$$\Delta = \frac{(3V)^8}{r r_1 \ldots r_7},$$

et Δ réprésente la norme de $\pm x \pm y \pm z \pm t$. (Voir un memoire de Mr. Sylvester dans le même cahier).* On a aussi

$$\Delta = \begin{vmatrix} . & . & . & . & x & y & z & t \\ . & . & x & y & . & . & t & z \\ . & x & . & z & . & t & . & y \\ . & y & z & . & t & . & . & x \\ x & . & . & t & . & z & y & . \\ y & . & t & . & z & . & x & . \\ z & t & . & . & y & x & . & . \\ t & z & y & x & . & . & . & . \end{vmatrix}$$

Si en général on considère n éléments

$$x_1, x_2 \ldots x_n,$$

en posant $$X_n = x_1 + x_2 \ldots + x_n,$$

et en désignant par

$$| X_n(1, 2, \ldots m) |,$$

le produit des facteurs linéaires qu'on deduit de X_n en changeant les signes à m des éléments $x_1, x_2 \ldots x_n$; en aura pour n impair

$$X_n | X_n(1) | | X_n(1, 2) | \ldots \left| X_n\left(1, 2, 3 \ldots \frac{n-1}{2}\right) \right| = -\Delta \ldots (1),$$

et pour n pair

$$X_n | X_n(1) | | X_n(1, 2) | \ldots \left| X_n\left(1, 2 \ldots \frac{n-2}{2}\right) \right| \left| X_n\left(\bar{1}, 2, 3 \ldots \frac{n}{2}\right) \right| = \Delta \quad \ldots\ldots\ldots (2),$$

où le symbole $$| X_n(\bar{1}, 2 \ldots \tfrac{1}{2}n) |$$

dénote que dans ce produit l'élément x_1 entre toujours parmi les éléments auxquels on a changé de signe. Le déterminant Δ résultera de la multiplication successive de l'équation $X_n = 0$ par l'unité, et par chacune des combinaisons deux à deux, quatre à quatre ... $(n-1)$ à $(n-1)$ si n est impair ; n à n si n est pair, des éléments $x_1, x_2 \ldots x_n$.

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* On the Relation between the Volume of a Tetrahedron and the Product of the Sixteen Algebraical Values of its Superficies. May 1853.

Il est évident que le nombre des facteurs linéaires du produit (1) sera

$$1 + n + \frac{n(n-1)}{2} + \ldots + \frac{n(n-1)\ldots\dfrac{(n-3)}{2}}{1.2\ldots\dfrac{n-1}{2}} = 2^{n-1},$$

et le nombre des facteurs du produit (2) sera

$$1+n+\frac{n(n-1)}{2}+\ldots+\frac{n(n-1)\ldots\left(\frac{n}{2}+2\right)}{1.2\ldots\left(\frac{n}{2}-1\right)}+\frac{1}{2}\,\frac{n(n-1)\ldots\left(\frac{n}{2}+1\right)}{1.2\ldots\frac{n}{2}} = 2^{n-1},$$

et que le nombre des équations qui découlent de l'équation $X_n = 0$ sera, soit pour n impair, soit pour n pair

$$1 + \frac{n(n-1)}{2} + \ldots + \frac{n(n-1)\ldots 3.2.1}{1.2.3\ldots(n-1)} = 2^{n-1}.$$

Il importe d'observer que, en supposant que les éléments $x_1, x_2 \ldots$ soient des radicaux quadratiques, le produit Δ des fonctions linéaires de ces radicaux est rationnel.

Si l'on désigne par $1, \alpha, \beta$ les trois racines de l'équation $w^3 - 1 = 0$, les expressions

$$x + y, \quad x + \alpha y, \quad x + \beta y,$$

sont les seules essentiellement différentes qu'on peut former en combinant deux éléments x, y. Supposons

$$x + y = 0, \quad x + \alpha y = 0, \quad x + \beta y = 0,$$

$$x^3 = a, \quad y^3 = b.$$

En multipliant chacune des équations supérieures par $1, x^2y, xy^2$, on aura

$$\Delta = \begin{vmatrix} . & 1 & 1 \\ 1 & . & a \\ 1 & b & . \end{vmatrix} = 0, \quad \Delta_1 = \begin{vmatrix} . & 1 & \alpha \\ \alpha & . & a \\ 1 & \alpha b & . \end{vmatrix} = 0, \quad \Delta_2 = \begin{vmatrix} . & 1 & \beta \\ \beta & . & a \\ 1 & \beta b & . \end{vmatrix} = 0,$$

et puisque Δ_1 se réduit identique à Δ en multipliant la dernière signe, et la seconde colonne par α, et en divisant la première signe et la première colonne par α; et analoguement en réduit $\Delta_2 = \Delta$ on aura

$$(x + y)(x + \alpha y)(x + \beta y) = \Delta.$$

Trois éléments x, y, z donnent neuf expressions différentes,

$$\left|\begin{array}{lll} x+y+z & x+y+\alpha z & x+y+\beta z \\ x+\alpha y+\beta z & x+\alpha y+z & x+\beta y+z \\ x+\beta y+\alpha z & \alpha x+y+z & \beta x+y+z \end{array}\right| \quad \ldots\ldots (3),$$

chacune desquelles égalée à zéro, et multipliée par $1, z^2y, z^2x, y^2z, y^2x, x^2z, x^2y, x^2y^2z^2, xyz$ donnera

$$\Delta = \left|\begin{array}{ccccccccc} . & . & . & . & . & . & 1 & 1 & 1 \\ . & . & 1 & . & . & 1 & . & c & . \\ . & . & 1 & . & 1 & . & c & . & . \\ . & 1 & . & . & . & 1 & . & . & b \\ . & 1 & . & 1 & . & . & b & . & . \\ 1 & . & . & . & 1 & . & . & . & a \\ 1 & . & . & 1 & . & . & . & a & . \\ . & . & . & c & b & a & . & . & . \\ 1 & 1 & 1 & . & . & . & . & . & . \end{array}\right| = 0,$$

ayant posé

$$x^3 = a, \quad y^3 = b, \quad z^3 = c;$$

et le produit des expressions (3) egalera $\pm\Delta$. La valeur de Δ est remarquable pour sa forme; en effet on a

$$\Delta = (a+b+c)^3 - (3abc)^3,$$

et il est remarquable aussi que les produits trois a trois des expressions (3) soit

$$a+b+c-3xyz,$$

$$\alpha(a+b+c)-3\beta xyz,$$

$$\beta(a+b+c)-3\alpha xyz.$$

En considérant quatre éléments x, y, z, t on aura vingt-sept expressions; c'est a dire l'expression $x+y+z+t$; douze expressions analogues à la $x+y+\alpha z+\beta t$; les six

$$\left|\begin{array}{ll} x+y+\alpha z+\alpha t & x+y+\beta z+\beta t \\ x+\alpha y+z+\alpha t & x+\beta y+z+\beta t \\ x+\alpha y+\alpha z+t & x+\beta y+\beta z+t \end{array}\right|$$

et les huit analogues aux deux

$$x+y+z+\alpha t, \quad x+y+z+\beta t.$$

Ces expressions egalées à zéro, et multipliées respectivement

par les vingt-sept quantités

$$1,\ t^2z\ldots,\ y^2z^2t^2\ldots,\ yzt\ldots,\ xy^2z^2t^2\ldots,\ x^2y^2z^2t^2,\ xyzt,$$

donneront par l'élimination un déterminant qui sera égal au produit des vingt-sept expressions supérieures.

En général le nombre des expressions sera 3^{n-1} pour n éléments ; et l'on aura 3^{n-1} quantités par lesquelles on doit multiplier les équations qui en découlent ; et par conséquent le déterminant sera de l'ordre 3^{n-1}. Et si les éléments seront des radicaux cubiques leur produit sera rationnel. Cette loi s'étend évidemment aux produit d'un nombre déterminé de radicaux de l'ennième ordre. Nous ajouterons seulement le cas du produit des quatre expressions qu'on obtient en combinant deux éléments avec les quatre racines de l'equation $w^4 - 1 = 0$. En désignant ces racines avec $1, \alpha, \beta, \gamma$ on aura les équations,

$$x + y = 0,\quad x + \alpha y = 0,\quad x + \beta y = 0,\quad x + \gamma y = 0\ ;$$

chacune desquelles multipliée par $1, xy^3, x^2y, x^2y^2$ donne

$$\Delta = \begin{vmatrix} . & . & 1 & 1 \\ 1 & . & b & . \\ . & 1 & . & a \\ 1 & 1 & . & . \end{vmatrix} = 0,$$

en posant $x^4 = a$, $y^4 = b$; et par conséquent

$$(x + y)(x + \alpha y)(x + \beta y)(x + \gamma y) = -\Delta,$$

et on soit que si x, y étaient des radicaux biquadratiques, le produit serait rationnel.

Nous ajouterons une dernière remarque sur une propriété des sommes des puissances égales de expressions supérieures. Par exemple le trinome

$$(x + y)^2 + \beta(x + \alpha y)^2 + \alpha(x + \beta y)^2 = H,$$

qui evidémment s'annule pour $x = 0$, $y = 0$ sera divisible par le produit xy ; et le quotient sera 3, 2 ; parceque le produit xy est répété trois fois, et chaque fois avec le coefficient 2. Par conséquent sera

$$H = 3.2xy.$$

Analoguement on peut démontrer l'équation

$$\begin{aligned}&(x + y + z)^3 + (x + \alpha y + \beta z)^3 + (x + \beta y + \alpha z)^3 \\ &+ \beta\{(x + y + \alpha z)^3 + (x + \alpha y + z)^3 + (\alpha x + y + z)^3\} \\ &+ \alpha\{(x + y + \beta z)^3 + (x + \beta y + z)^3 + (\beta x + y + z)^3\} = 3^3.2.3xyz,\end{aligned}$$

et ainsi de suite. Ces relations sont analogues à quelques-unes de celles que M. Cauchy* a trouvées en considérant seulement les deux racines de l'équation $w^2 - 1 = 0$.

NOTE.—PROB. 7.† Si $\alpha = 0$, $\beta = 0$, $\gamma = 0$ sont les équations des trois côtès d'un triangle l, m, n, trois indéterminées on a

$$\begin{vmatrix} . & \alpha & \beta & \gamma \\ \alpha & . & n & m \\ \beta & n & . & l \\ \gamma & m & l & . \end{vmatrix} = 0,$$

pour l'équation d'une conique inscrite. En égard à ce qu'on a démontré supérieurment cette équation pourra s'écrire

$$\pm (l\alpha)^{\frac{1}{2}} \pm (m\beta)^{\frac{1}{2}} \pm (n\gamma)^{\frac{1}{2}} = 0.$$

Pour déterminer l, m, n, j'observe que l'ellipse *maximum* inscrite dans un triangle touche les milieux de ces côtés. Or les équations des droites qui unissant deux à deux les points de contact étant en général

$$m\beta + n\gamma - l\alpha = 0, \quad n\gamma + l\alpha - m\beta = 0, \quad l\alpha + m\beta - n\gamma = 0;$$

et dans ce cas ces droites étant respectivement parallèles aux côtés du triangle, on aura les équations

$$m \sin(\alpha - \beta) + n \sin(\alpha - \gamma) = 0,$$

$$n \sin(\beta - \gamma) + l \sin(\beta - \alpha) = 0,$$

$$l \sin(\gamma - \alpha) + m \sin(\gamma - \beta) = 0,$$

d'où

$$l : m : n = a : b : c,$$

a, b, c étant les longueurs des trois côtés. Par conséquent l'équation de l'ellipse *maximum* inscrite sera

$$\pm (a\alpha)^{\frac{1}{2}} \pm (b\beta)^{\frac{1}{2}} \pm (c\gamma)^{\frac{1}{2}} = 0.$$

On trouvera l'équation de l'ellipse *minimum* circonscrite en observant qu'elle est concentrique et semblable à l'ellipse *maximum* inscrite.

Pavie, le 19 *Juillet*, 1853.

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* *Exercises d'Analyse et de Physique Mathématiques*, tom. II. An. 1841.
† *Journal, May* 1853.

ON A PARTICULAR CASE OF THE DESCENT OF A HEAVY BODY IN A RESISTING MEDIUM.

By J. C. Maxwell.

Every one must have observed that when a slip of paper falls through the air, its motion, though undecided and wavering at first, sometimes becomes regular. Its general path is not in the vertical direction, but inclined to it at an angle which remains nearly constant, and its fluttering appearance will be found to be due to a rapid rotation round a horizontal axis. The direction of deviation from the vertical depends on the direction of rotation.

If the positive directions of an axis be toward the right hand and upwards, and the positive angular direction opposite to the direction of motion of the hands of a watch, then, if the rotation is in the positive direction, the horizontal part of the mean motion will be positive.

These effects are commonly attributed to some accidental peculiarity in the form of the paper, but a few experiments with a rectangular slip of paper (about two inches long and one broad), will shew that the direction of rotation is determined, not by the irregularities of the paper, but by the initial circumstances of projection, and that the symmetry of the form of the paper greatly increases the distinctness of the phenomena. We may therefore assume that if the form of the body were accurately that of a plane rectangle, the same effects would be produced.

The following investigation is intended as a general explanation of the true cause of the phenomenon.

I suppose the resistance of the air caused by the motion of the plane to be in the direction of the normal and to vary as the square of the velocity estimated in that direction.

Now though this may be taken as a sufficiently near approximation to the magnitude of the resisting force on the plane taken as a whole, the pressure on any given element of the surface will vary with its position so that the resultant force will not generally pass through the centre of gravity.

It is found by experiment that the position of the centre of pressure depends on the tangential part of the motion, that it lies on that side of the centre of gravity towards which the tangential motion of the plane is directed, and that its distance from that point increases as the tangential velocity increases.

I am not aware of any mathematical investigation of this effect. The explanation may be deduced from experiment.

Place a body similar in shape to the slip of paper obliquely in a current of some visible fluid. Call the edge where the fluid first meets the plane the first edge, and the edge where it leaves the plane, the second edge, then we may observe that

(1) On the anterior side of the plane the velocity of the fluid increases as it moves along the surface from the first to the second edge, and therefore by a known law in hydrodynamics, the pressure must diminish from the first to the second edge.

(2) The motion of the fluid behind the plane is very unsteady, but may be observed to consist of a series of eddies diminishing in rapidity as they pass behind the plane from the first to the second edge, and therefore relieving the posterior pressure most at the first edge.

Both these causes tend to make the total resistance greatest at the first edge, and therefore to bring the centre of pressure nearest to that edge.

Hence the moment of the resistance about the centre of gravity will always tend to turn the plane towards a position perpendicular to the direction of the current, or, in the case of the slip of paper, to the path of the body itself. It will be shewn that it is this moment that maintains the rotatory motion of the falling paper.

When the plane has a motion of rotation, the resistance will be modified on account of the unequal velocities of different parts of the surface. The magnitude of the whole resistance at any instant will not be sensibly altered if the velocity of any point due to angular motion be small compared with that due to the motion of the centre of gravity. But there will be an additional moment of the resistance round the centre of gravity, which will always act in the direction opposite to that of rotation, and will vary directly as the normal and angular velocities together.

The part of the moment due to the obliquity of the motion will remain nearly the same as before.

We are now prepared to give a general explanation of the motion of the slip of paper after it has become regular.

Let the angular position of the paper be determined by the angle between the normal to its surface and the axis of x, and let the angular motion be such that the normal, at first coinciding with the axis of x, passes towards that of y.

The motion, speaking roughly, is one of descent, that is, in the negative direction along the axis of y.

The resolved part of the resistance in the vertical direction will always act upwards, being greatest when the plane of the paper is horizontal, and vanishing when it is vertical.

When the motion has become regular, the effect of this force during a whole revolution will be equal and opposite to that of gravity during the same time.

Since the resisting force increases while the normal is in its first and third quadrants, and diminishes when it is in its second and fourth, the maxima of velocity will occur when the normal is in its first and third quadrants, and the minima when it is in the second and fourth.

The resolved part of the resistance in the horizontal direction will act in the positive direction along the axis of x in the first and third quadrants, and in the negative direction during the second and fourth; but since the resistance increases with the velocity, the whole effect during the first and third quadrants will be greater than the whole effect during the second and fourth. Hence the horizontal part of the resistance will act on the whole in the positive direction, and will therefore cause the general path of the body to incline in that direction, that is, toward the right.

That part of the moment of the resistance about the centre of gravity which depends on the angular velocity will vary in magnitude, but will always act in the negative direction. The other part, which depends on the obliquity of the plane of the paper to the direction of motion, will be positive in the first and third quadrants and negative in the second and fourth; but as its magnitude increases with the velocity, the positive effect will be greater than the negative.

When the motion has become regular, the effect of this excess in the positive direction will be equal and opposite to the negative effect due to the angular velocity during a whole revolution.

The motion will then consist of a succession of equal and similar parts performed in the same manner, each part corresponding to half a revolution of the paper.

These considerations will serve to explain the lateral motion of the paper, and the maintenance of the rotatory motion.

Similar reasoning will shew that whatever be the initial motion of the paper, it cannot remain uniform.

Any accidental oscillations will increase till their amplitude exceeds half a revolution. The motion will then become

one of rotation, and will continually approximate to that which we have just considered.

It may be also shewn that this motion will be unstable unless it take place about the longer axis of the rectangle.

If this axis is inclined to the horizon, or if one end of the slip of paper be different from the other, the path will not be straight, but in the form of a helix. There will be no other essential difference between this case and that of the symmetrical arrangement.

Trinity College, April 5, 1853.

ON THE RELATION BETWEEN DIFFERENT CURVES AND CONES CONNECTED WITH A SERIES OF CONFOCAL ELLIPSOIDS.

By John Y. Rutledge, A.M., Trinity College, Dublin.

No. II.

In a preceding number of this *Journal* we established a natural connection between a very interesting and important system of curves traced upon the surface of an ellipsoid; or, more generally, traced upon any central surface of the second order. The present article seeks to complete that investigation, and by some additional examples, not perhaps devoid of interest, still farther to illustrate the singular relations which exist in this portion of the geometry of surfaces of the second order. Let the equation of the ellipsoid referred to its centre be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

We have already seen that, an ellipsoid being given, we are given with it an infinite set of hyperboloids—each set consisting of three—which, when viewed through the medium of corresponding points, are fixed with reference to an infinite series of confocal ellipsoids and from which result the different curves and cones in question for each ellipsoid. In the present instance let the equation of one of the fixed hyperboloids be

$$\frac{x^2}{a^2-\rho^2}+\frac{y^2}{b^2-\rho^2}+\frac{z^2}{c^2-\rho^2}=1,$$

ρ being (according to definition) a semidiameter of the ellipsoid. This surface will intersect the ellipsoid in a line of

curvature. Its asymptotic cone will also intersect the ellipsoid in a curve, which, for convenience sake, we purpose calling the 'asymptotic correlative' of the line of curvature. This asymptotic cone will intersect the surface of elasticity in an analogous curve: now any curve being given on the surface of elasticity, we can readily obtain its conjugate on the ellipsoid; this conjugate, in the present instance, we propose calling the 'correlative' of the line of curvature. We consequently see that, any line of curvature being given, we are given at the same time two correlative curves, which we may always discuss in connection with it. The equations of the respective curves are

$$\left.\begin{aligned} \frac{x^2}{a^2-\rho^2} + \frac{y^2}{b^2-\rho^2} + \frac{z^2}{c^2-\rho^2} &= 0, \\ \frac{x^2}{a^2(a^2-\rho^2)} + \frac{y^2}{b^2(b^2-\rho^2)} + \frac{z^2}{c^2(c^2-\rho^2)} &= 0, \\ \frac{x^2}{a^4(a^2-\rho^2)} + \frac{y^2}{b^4(b^2-\rho^2)} + \frac{z^2}{c^4(c^2-\rho^2)} &= 0, \end{aligned}\right\} \quad \ldots\ldots(1).$$

The first being the equation of the 'asymptotic correlative' line; the second of the line of curvature; and the third of the line 'correlative' of the line of curvature.

Now let the complete differential equation of any surface referred to ordinary rectangular coordinates, be

$$Pdx + Qdy + Rdz = 0,$$

making $$X = \frac{P}{R} \text{ and } Y = \frac{Q}{R}.$$

Geometers have called X and Y the normal coordinates of any point upon the surface; and it has been remarked, that X and Y being given, the point upon the surface may be determined by the equations X, Y, and the equation of the surface $F(x, y, z) = 0$.

It is easy to see that the equations of the preceding curves, expressed in normal coordinates, are

$$\left.\begin{aligned} (\xi)\ \frac{a^4X^2}{a^2-\rho^2} + \frac{b^4Y^2}{b^2-\rho^2} + \frac{c^4}{c^2-\rho^2} &= 0, \\ (\eta)\ \frac{a^2X^2}{a^2-\rho^2} + \frac{b^2Y^2}{b^2-\rho^2} + \frac{c^2}{c^2-\rho^2} &= 0, \\ (\zeta)\ \frac{X^2}{a^2-\rho^2} + \frac{Y^2}{b^2-\rho^2} + \frac{1}{c^2-\rho^2} &= 0, \end{aligned}\right\} \quad \ldots\ldots(2).$$

These are now the equations of sphero-conics traced upon a sphere whose radius is unity, by tangent-planes parallel to the planes tangent to the ellipsoid along the respective curves under consideration; (ξ) is therefore the sphero-conic derived from the asymptotic correlative line, (η) the sphero-conic derived from the line of curvature, and (ζ) the sphero-conic derived from the correlative line of the line of curvature. From the mere fact of these curves being of the second order, it follows that a tangent plane at any point makes angles whose sum or difference is constant with two fixed planes, viz. the planes tangent to the sphere at their respective foci. It is well known that the equation of a sphero-conic, referred to its principal diametral arcs for arcs of reference, is

$$\frac{X^2}{\tan^2\alpha} + \frac{Y^2}{\tan^2\beta} = 1,$$

where α and β denote the distances from the origin of the points in which the curve meets the X and Y arcs of reference. The foci are of course determined by the equation

$$\tan^2\gamma = \frac{\tan^2\alpha - \tan^2\beta}{1 + \tan^2\beta},$$

and the cyclic arcs by the equations

$$\cos^2\phi = \frac{\tan^2\gamma}{\tan^2\alpha}, \quad \sin^2\phi = \frac{\sin^2\beta}{\sin^2\alpha},$$

where ϕ denotes the angle between one of the cyclic arcs and the greatest diameter. Let us now apply these expressions to equations (2), and we shall have for (ξ)

$$\tan^2\alpha = \frac{c^4(a^2-\rho^2)}{a^4(\rho^2-c^2)}, \qquad \tan^2\beta = \frac{c^4(b^2-\rho^2)}{b^4(\rho^2-c^2)};$$

for (η)

$$\tan^2\alpha' = \frac{c^2}{a^2}\cdot\frac{a^2-\rho^2}{\rho^2-c^2}, \qquad \tan^2\beta' = \frac{c^2}{b^2}\cdot\frac{(b^2-\rho^2)}{(\rho^2-c^2)};$$

and for (ζ)

$$\tan^2\alpha'' = \frac{a^2-\rho^2}{\rho^2-c^2}, \qquad \tan^2\beta'' = \frac{b^2-\rho^2}{\rho^2-c^2}.$$

We consequently have the curious relations

$$\tan\alpha.\tan\alpha'' = \tan^2\alpha', \quad \tan\beta.\tan\beta'' = \tan^2\beta';$$

i.e. the tangents of the arcs (α, α', &c.) are in continued proportion. We also have

$$\frac{\tan\alpha}{\tan\alpha''} = \frac{c^2}{a^2}, \quad \frac{\tan\alpha'}{\tan\alpha''} = \frac{c}{a}, \quad \frac{\tan\beta}{\tan\beta''} = \frac{c^2}{b^2}, \quad \frac{\tan\beta'}{\tan\beta''} = \frac{c}{b}.$$

The ratios, therefore, of the tangents of (α, α', &c.) are independent of the semi-diameter of the ellipsoid (ρ), and are identical for all the derived curves of the form (2) traced upon the surface of the sphere. Again, for (ξ), we have

$$\tan^2\gamma = \frac{c^4}{a^4} \cdot \frac{a^2 - b^2}{b^2 - c^2} \cdot \left\{\frac{a^2b^2 - \rho^2(a^2 + b^2)}{b^2c^2 - \rho^2(b^2 + c^2)}\right\},$$

for (η)

$$\tan^2\gamma' = \frac{c^2}{a^2} \cdot \frac{a^2 - b^2}{b^2 - c^2},$$

and for (ζ)

$$\tan^2\gamma'' = \frac{a^2 - b^2}{b^2 - c^2};$$

we consequently have

$$\frac{\tan\gamma'}{\tan\gamma''} = \frac{\tan\alpha'}{\tan\alpha''};$$

and since $\tan\gamma'$ and $\tan\gamma''$ are independent of (ρ), it follows "that all the curves of the sphere corresponding to the lines of curvature on the ellipsoid are biconfocal," the same theorem holding also for the spherical curves corresponding to the lines 'correlative' of the line of curvature. We also perceive that the two last-mentioned conics of the sphere are, for any given value of (ρ), biconcyclic. From the values found for $\tan\gamma$, $\tan\gamma'$, and $\tan\gamma''$ it follows, that the foci of the conic (ξ) correspond to the points upon the ellipsoid, at which the tangent planes are parallel to the circular sections (of the cone) of the line of constant curvature; *i.e.* the curve-locus of the points at which the measure of curvature of the ellipsoid has a constant value. The equation of this curve, as has been shewn in the paper to which we have already referred, is

$$\frac{a^2 - \rho^2}{a^4} \cdot x^2 + \frac{b^2 - \rho^2}{b^4} \cdot y^2 + \frac{c^2 - \rho^2}{c^4} \cdot z^2 = 0,$$

where (ρ) denotes the constant perpendicular from the centre upon the tangent plane to the ellipsoid at any point upon the curve. The tangent planes to the ellipsoid, therefore, along the asymptotic correlative line of the line of curvature, make angles whose sum or difference is constant with the circular sections of the corresponding line of constant curvature. The foci of the curve (η) evidently correspond to the umbilics of the ellipsoid; we have therefore a proof of the well-known theorem, "*The tangent-planes to the ellipsoid along a line of curvature make angles whose sum*

or difference is constant with the planes of circular section." This pleasing proof was first given by Professor Graves. The foci of the conic (ζ) correspond to the points upon the ellipsoid at which the tangent-planes are normal to the asymptotes of the focal hyperbola, and consequently the tangent-planes to the ellipsoid along the line correlative of the line of curvature make angles whose sum or difference is constant with the planes normal to the asymptotes of the focal hyperbola. We also evidently have the interesting theorem, "*The curves upon the sphere which correspond to the lines correlative of the line of curvature, when considered upon an infinite series of confocal ellipsoids, are all biconfocal.*" From what has been stated it is easy to see that the tangent-planes to the ellipsoid along the asymptotic correlative of the line of curvature form by their ultimate intersections a developable surface, of which the 'arêtes' are parallel to the sides of a central cone (*i.e.* which has its vertex at the centre) intersecting the surface of the ellipsoid in the line of constant curvature. Also, as is well known, the tangent-planes to the ellipsoid, along the line of curvature, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central cone intersecting the surface of the ellipsoid in a sphero-conic. In like manner, the tangent-planes to the ellipsoid, along the line correlative of the line of curvature, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central cone that intersects the surface of the ellipsoid in a curve, which in our former paper we have named the curve of 'reference' of the sphero-conic. The last-mentioned cone we have also shewn to be the reciprocal of the asymptotic cone of the fixed hyperboloid. Hence follows the curious theorem—"*The tangent-plane to the ellipsoid at any point upon the line correlative to the line of curvature is normal to a side of the corresponding asymptotic cone;*" or, in other words, that the different normals to the ellipsoid along the line correlative of the line of curvature are parallel to the sides of the corresponding asymptotic cone. The equations of this new system of curves are

$$\left.\begin{aligned} \frac{a^2-\rho^2}{a^4}x^2+\frac{b^2-\rho^2}{b^4}y^2+\frac{c^2-\rho^2}{c^4}z^2=0 \\ \frac{a^2-\rho^2}{a^2}x^2+\frac{b^2-\rho^2}{b^2}y^2+\frac{c^2-\rho^2}{c^2}z^2=0 \\ (a^2-\rho^2)x^2+(b^2-\rho^2)y^2+(c^2-\rho^2)z^2=0 \end{aligned}\right\} \quad \ldots\ldots\ldots(3).$$

The first being the equation of the line of constant curvature, the second of the sphero-conic, and the third of the sphero-conic's curve of reference. The equations of the corresponding curves upon the sphere whose radius is unity are

$$\left.\begin{aligned}
&(\xi_1)\ (a^2-\rho^2)X^2+(b^2-\rho^2)Y^2+(c^2-\rho^2)=0\\
&(\eta_1)\ a^2(a^2-\rho^2)X^2+b^2(b^2-\rho^2)Y^2+c^2(c^2-\rho^2)=0\\
&(\zeta_1)\ a^4(a^2-\rho^2)X^2+b^4(b^2-\rho^2)Y^2+c^4(c^2-\rho^2)=0
\end{aligned}\right\}\ \ldots(4).$$

The first is the equation of the conic corresponding to the line of constant curvature, the second of the conic corresponding to the sphero-conic, and the third of the conic corresponding to the sphero-conic's curve of reference.

For the equation (ξ_1) we have

$$\tan^2\alpha_1=\frac{\rho^2-c^2}{a^2-\rho^2},\qquad \tan^2\beta_1=\frac{\rho^2-c^2}{b^2-\rho^2};$$

for (η_1) we have

$$\tan^2\alpha_1'=\frac{c^2}{a^2}\cdot\frac{\rho^2-c^2}{a^2-\rho^2},\qquad \tan^2\beta_1'=\frac{c^2}{b^2}\cdot\frac{\rho^2-c^2}{b^2-\rho^2};$$

and for (ζ_1) we have

$$\tan^2\alpha_1''=\frac{c^4}{a^4}\cdot\frac{\rho^2-c^2}{a^2-\rho^2},\qquad \tan^2\beta_1''=\frac{c^4}{b^4}\cdot\frac{\rho^2-c^2}{b^2-\rho^2}.$$

From which we obtain the following relations,

$$\tan\alpha_1\tan\alpha_1''=\tan\alpha_1'^2,\quad \tan\beta_1\tan\beta_1''=\tan\beta_1'^2.$$

So that the tangents of the arcs (α_1, α_1', &c.) are in continued proportion. Also

$$\tan\alpha.\tan\alpha_1=\tan\alpha'.\tan\alpha_1'=\tan\alpha''.\tan\alpha_1''=\frac{c^2}{a^2},$$

$$\tan\beta.\tan\beta_1=\tan\beta'.\tan\beta_1'=\tan\beta''.\tan\beta_1''=\frac{c^2}{b^2},$$

$$\tan^2\alpha_1\tan^2\alpha''=\tan^2\beta_1\tan^2\beta''=1.$$

We in like manner perceive that the ratios

$$\frac{\tan\alpha_1}{\tan\alpha_1''}=\frac{a^2}{c^2},\quad \frac{\tan\alpha_1'}{\tan\alpha_1''}=\frac{a}{c},\quad \frac{\tan\beta_1}{\tan\beta_1''}=\frac{b^2}{c^2},\quad \frac{\tan\beta_1'}{\tan\beta_1''}=\frac{b}{c},$$

are the reciprocals of the analogous ratios which we have already found for the equations (2). The foci of the conics

(ξ_1, η_1, ζ_1) are determined by the equations

$$\tan^2\gamma_1 = \frac{c^2-\rho^2}{a^2-\rho^2}\cdot\frac{a^2-b^2}{b^2-c^2}, \quad \tan^2\gamma_1' = \frac{c^2}{a^2}\cdot\frac{c^2-\rho^2}{a^2-\rho^2}\cdot\frac{a^2-b^2}{b^2-c^2}\cdot\frac{a^2+b^2-\rho^2}{b^2+c^2-\rho^2},$$

$$\tan^2\gamma_1'' = \frac{c^4}{a^4}\cdot\frac{c^2-\rho^2}{a^2-\rho^2}\cdot\frac{a^2-b^2}{b^2-c^2}\cdot\frac{a^4+a^2b^2+b^4-\rho^2(a^2+b^2)}{b^4+b^2c^2+c^4-\rho^2(b^2+c^2)};$$

we also have $\cos^2\phi_1 = -\tan^2\gamma''$.

The preceding relations between the system of six conics upon the auxiliary sphere of radius unity are sufficiently interesting to repay attention. Since the preceding values for $\tan\gamma_1$, $\tan\gamma_1'$, &c. all include (ρ), we perceive that no one of the conics represented by equations (4) is biconfocal for the system of conics derived from the corresponding system of curves upon the surface of the ellipsoid obtained by the variation of (ρ). We at the same time, however, perceive that the sphero-conics (ζ, ξ_1) are identical for their corresponding curves derived upon an infinite series of confocal ellipsoids from one and the same fixed hyperboloid. The conics (ζ) and (ξ_1) are in fact the reciprocals one of the other; *i.e.* the equation of one being

$$\frac{X^2}{\tan^2\alpha} + \frac{Y^2}{\tan^2\beta} = 1,$$

the equation of the other is

$$\tan^2\alpha X^2 + \tan^2\beta Y^2 = 1.$$

Let us now return to the equations (3) on the surface of the ellipsoid, and, as we have stated in our former paper, we can easily prove "*the tangent planes to the ellipsoid, along the line of constant curvature, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central cone,* viz. *the asymptotic cone of the fixed hyperboloid; consequently, each 'arête' makes angles whose sum or difference is constant with the asymptotes of the focal hyperbola.*" Also, each tangent plane to the ellipsoid, along the line of constant curvature, makes angles whose sum or difference is constant with the circular sections of the fixed hyperboloid. Again, the tangent planes to the ellipsoid, along the sphero-conic, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central cone intersecting the ellipsoid in a line of curvature. The tangent planes, also to the ellipsoid, along the curve of reference of the sphero-

conic, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central cone intersecting the ellipsoid in the line correlative of the line of curvature. If we look for the focal lines of the last-mentioned cone, we can easily see that the tangent planes to the ellipsoid at the points in which the focal lines pierce the surface, make with the axis of (x) angles, the tangents of which bear a constant ratio to the tangents of the angles $+\gamma_1''$ or $-\gamma_1''$ for the infinite series of confocal ellipsoids; the curve on each ellipsoid being of course considered as derived from one and the same fixed hyperboloid. An analogous theorem holds for the tangents of the angles made with the axis of (x) by the planes tangent to the ellipsoid at the points in which the focal lines of the cone of constant curvature pierce its surface and the tangents of the angles $+\gamma$ or $-\gamma$. Let the tangent planes in the first instance make with the axis of (x) the angle $(\pm\theta')$, in the second instance the angle $(\pm\theta)$; we shall then have

$$\frac{\tan^2\theta}{\tan^2\gamma} = \frac{\tan^2\gamma_1''}{\tan^2\theta'} = \frac{c^2-\rho^2}{a^2-\rho^2}.$$

Let $(x'y'z')$ denote any point upon the sphero-conic, traced upon the surface of the ellipsoid by the sphere-cone of radius (ρ); then if we write

$$\frac{x'}{a} = \cos i, \quad \frac{y'}{b} = \cos i_1, \quad \frac{z'}{c} = \cos i_{11},$$

we shall have determined a corresponding side of the cone of reference, such that if a plane be drawn normal to their side and tangent to the ellipsoid, the perpendicular intercept measured from the centre will equal (ρ), and the point of contact of the tangent plane with the ellipsoid will of course be a point upon the line of constant curvature. Any side of any one of the three preceding cones being given, we have indicated in our former paper a geometrical construction, by means of the ellipsoid of reference, to determine the two corresponding sides upon the two remaining cones; the ellipsoid of reference being such that the original given ellipsoid is with respect to it the reciprocal polar of the auxiliary sphere of radius unity. Similarly, let (x_1, y_1, z_1) be the coordinates of a point upon the line of curvature; if we then refer the radius vector drawn to this point to its line of reference, we shall obtain a side of the fixed asymptotic cone; and a tangent plane drawn normal to this side will touch the ellipsoid in a point upon the curve

correlative to the line of curvature. The same geometrical construction to which we have alluded, by means of the ellipsoid of reference, will enable us, any side of any one of the three preceding cones being given, to determine the two corresponding sides upon the two remaining cones.

So far we perceive that, resulting from one and the same fixed hyperboloid, *there are upon the surface of the ellipsoid six curves and six cones which have a natural and intimate connection;* while at the same time there are six derived sphero-conics upon the surface of the auxiliary sphere of radius unity. We next perceive that, since when viewed through the medium of corresponding points, the asymptotic cone is fixed with reference to an infinite series of confocal ellipsoids, as also its reciprocal cone, the cone of reference of the equiradial cone of radius (ρ), *there are two systems of parallel developable surfaces which envelope the entire series of confocal ellipsoids;* viz. *the developable surface along the line of constant curvature, and the developable along the line correlative of the line of curvature.* In fact, when the ellipsoid degenerates into a point, the first-named developable surface degenerates into the asymptotic cone, and the second into the asymptotic cone's reciprocal or supplementary cone. Let the ellipsoid be one of revolution round its mean axis suppose, and we shall have for the developable surface, circumscribed to the ellipsoid along the line of constant curvature, the equation

$$\frac{x^2}{a^2-\rho^2}+\frac{y^2}{b^2-\rho^2}+\frac{z^2}{a^2-\rho^2}-2y\rho\frac{\sqrt{(a^2-b^2)}}{(b^2-\rho^2)\sqrt{(a^2-\rho^2)}}=\frac{\rho^2(a^2-b^2)}{(a^2-\rho^2)(\rho^2-b^2)};$$

and for the developable surface circumscribed along the line correlative of the line of curvature, the equation

$$(a^2-\rho^2)x^2+(b^2-\rho^2)y^2+(a^2-\rho^2)z^2-2y\sqrt{(\rho^2-b^2)}\sqrt{\{a^2(a^2-\rho^2)-b^2(b^2-\rho^2)\}}$$
$$=a^2(a^2-\rho^2)-b^2(b^2-\rho^2).$$

Now the tangent planes to the fixed hyperboloid and at the same time normal to any one of the infinite series of confocal ellipsoids, will trace upon that one by the normal points, the geodesic line or lines which touch the pair of opposite lines of curvature, made by the intersection of the fixed hyperboloid with the ellipsoid in question; while at the same time the developable surface circumscribed to the ellipsoid along the geodesic line, will have as locus to its 'arête de rebrousment,' the reciprocal polar of the fixed hyperboloid taken with respect to the ellipsoid circumscribed.

The equation of this locus surface will be of the form

$$\frac{a^2-\rho^2}{a^4}x^2+\frac{b^2-\rho^2}{b^4}y^2+\frac{c^2-\rho^2}{c^4}z^2=1,$$

the asymptotic cone of which we perceive to be the central cone which intersects the ellipsoid in the line of constant curvature. If we now consider the two semi-diameters of the ellipsoid conjugate with (ρ), we shall have two more fixed hyperboloids, the consideration of which will give us the previously enumerated curves and cones in sets of three, as well as the derived sphero-conics upon the auxiliary sphere of radius unity. In our former paper we have stated the curious relations which connect any two of these three fixed hyperboloids with the third; it is needless therefore to repeat them here.

Professor Chasles, in the notes to his admirable history of Geometry, has demonstrated the very beautiful theorem, "*Given in magnitude and position any three conjugate semi-diameters of an ellipsoid, construct the surface.*" This theorem in a remarkable manner completes our theory; for we see that if any three conjugate semi-diameters of an ellipsoid be given in magnitude and position, we can construct the surface; while at the same time we have three hyperboloids fixed with reference to an infinite series of confocal ellipsoids, the consideration of which gives on each ellipsoid of the series the remarkable and interesting system of curves that we have enumerated in the present paper. To avoid needless repetition, we have hitherto restricted ourselves to the consideration of the ellipsoid; the reader can, however, without difficulty extend the preceding theorems to all central surfaces of the second order. The modifications which they undergo in the case of the paraboloids may be also had, by remembering that instead of the semi-diameters (ρ), we must consider the parallel bifocal chords. Let us now consider the line of curvature

$$\frac{x^2}{a^2(a^2-\rho^2)}+\frac{y^2}{b^2(b^2-\rho^2)}+\frac{z^2}{c^2(c^2-\rho^2)}=0$$

upon the surface of the fixed hyperboloid. The equation of its derived curve upon the auxiliary sphere of radius unity then is

$$\frac{a^2-\rho^2}{a^2}X^2+\frac{b^2-\rho^2}{b^2}Y^2+\frac{c^2-\rho^2}{c^2}=0;$$

we consequently have

$$\tan^2\alpha = \frac{a^2}{c^2}\cdot\frac{(\rho^2-c^2)}{a^2-\rho^2}, \qquad \tan^2\beta = \frac{b^2}{c^2}\cdot\frac{\rho^2-c^2}{b^2-\rho^2},$$

$$\tan^2\gamma = \frac{c^2-\rho^2}{a^2-\rho^2}\cdot\frac{a^2-b^2}{b^2-c^2}, \qquad \cos^2\phi = -\frac{c^2}{a^2}\cdot\frac{a^2-b^2}{b^2-c^2};$$

from which we perceive that this curve is the reciprocal of the conic (η), and that it is biconfocal with the conic (ξ_1). From the value of $\cos\phi$ it is manifest that the conics derived from the lines of curvature upon the fixed hyperboloids, formed by the intersection of any one ellipsoid, are all biconcyclic; we have already seen that the conics derived from the lines of curvature, when considered upon the ellipsoid, are all biconfocal. The tangent planes to the fixed hyperboloid, along the line of curvature, form by their ultimate intersections a developable surface, the 'arêtes' of which are parallel to the sides of a central equiradial cone, the equation of which is

$$\frac{a^2x^2}{a^2-\rho^2}+\frac{b^2y^2}{b^2-\rho^2}+\frac{c^2z^2}{c^2-\rho^2}=0,$$

the reciprocal of the equiradial cone already found for the ellipsoid. The equation of its derived conic is

$$a^2(a^2-\rho^2)\,X^2+b^2(b^2-\rho^2)\,Y^2+c^2(c^2-\rho^2)=0,$$

which is identical with the equation (η_1), from which we perceive that the tangent planes to the ellipsoid and the fixed hyperboloid along their respective related sphero-conics are parallel, and make angles whose sum or difference is constant with the same pair of fixed planes.

Let two confocal hyperboloids, the equations of which are

$$(\phi)\ \frac{x^2}{\mu^2}+\frac{y^2}{\mu'^2}+\frac{z^2}{\mu''^2}=1, \quad (\omega)\ \frac{x^2}{\nu^2}+\frac{y^2}{\nu'^2}+\frac{z^2}{\nu''^2}=1,$$

intersect the surface of the ellipsoid in the point $(x'y'z')$, and in each surface let the right line of reference corresponding to the radius vector (ρ') be found; then, if we draw perpendicular to each a plane tangent to the surface to which it belongs, the points of contact will be corresponding points, *i.e.*

$$\frac{x_{,}}{a}=\frac{x_{,,}}{\mu}=\frac{x_{,,,}}{\nu},$$

where ($x_{,}$, $y_{,}$, &c., $x_{,,}$, $y_{,,}$, &c.) represent the points of contact on the respective surfaces. Let us next consider the cor-

responding points to (x', y', z') on the series of confocal ellipsoids. It is well known that the curve of intersection of ϕ and ω will be the locus of these corresponding points; consequently, if we repeat the preceding construction for each corresponding point on the several ellipsoids of the series, the points of contact of the several tangent planes to the surfaces ϕ and ω, will give on each a locus of corresponding points. The respective equations of these loci on the surfaces ϕ and ω, are

$$\frac{x^2}{\mu^4\nu^2} + \frac{y^2}{\mu'^4\nu'^2} + \frac{z^2}{\mu''^4\nu''^2} = 0, \quad \frac{x^2}{\nu^4\mu^2} + \frac{y^2}{\nu'^4\mu'^2} + \frac{z^2}{\nu''^4\mu''^2} = 0.$$

It is easy to see that these equations represent on each surface the lines correlative to the lines of curvature indicated by the equation

$$\frac{x^2}{\mu^2\nu^2} + \frac{y^2}{\mu'^2\nu'^2} + \frac{z^2}{\mu''^2\nu''^2} = 0.$$

This results from the fact that the asymptotic cone of one of two intersecting confocal surfaces of the second order is the cone of reference of the central cone, which passes through the line of curvature on the other formed by their intersection. In the ellipsoid, since the right line of reference of (ρ') lies at once on the asymptotic cones of ϕ and ω, it is evident that this right line must be one of the four sides of intersection of the two asymptotic cones; if then, perpendicular to either of these sides, we draw a plane tangent to the ellipsoid at the point $(x_i y_i z_i)$, and if A denote the double of the triangle formed by the perpendicular intercept from the centre, the radius vector (ρ_i) and its projection on the tangent plane, we shall have

$$A^2 = \mu^2\nu^2 - (\mu^2 + \nu'^2)(\mu^2 + \nu''^2).$$

So that this area is constant for the infinite series of confocal surfaces. This expression for an area, which in the theory of rotation has an important physical signification, will not be found in the solution of many questions devoid of utility.

Let $(r, r_{,}, r_{,,})$ be the angles formed by a side of intersection of the asymptotic cones with the axes of (x, y, z), and we shall have

$$\cos^2 r = \frac{\mu^2\nu^2}{(a^2 - b^2)(a^2 - c^2)}, \quad \cos^2 r_{,} = \frac{\mu'^2\nu'^2}{(b^2 - a^2)(b^2 - c^2)},$$

$$\cos^2 r_{,,} = \frac{\mu''^2\nu''^2}{(a^2 - c^2)(b^2 - c^2)}.$$

Since however, if with the point $(x'y'z')$ as centre we construct the three confocal surfaces intersecting in the centre of the original ellipsoid, the equations of which referred to the three normals at the point $(x'y'z')$ as axes of (ξ, η, ζ), are

$$(\varpi)\ \frac{\xi^2}{a^2}+\frac{\eta^2}{\mu^2}+\frac{\zeta^2}{\nu^2}=1,\quad (\varpi')\ \frac{\xi^2}{b^2}+\frac{\eta^2}{\mu''^2}+\frac{\zeta^2}{\nu''^2}=1,$$

$$(\varpi'')\ \frac{\xi^2}{c^2}+\frac{\eta^2}{\mu'''^2}+\frac{\zeta^2}{\nu'''^2}=1,$$

then the four sides of intersection of the two asymptotic cones of ϕ and ω are the four bifocal chords, which can be drawn from the centre of the original ellipsoid through the focal curves of the newly constructed system, we shall have, as I have elsewhere demonstrated,

$$\cos^2 r=\frac{x''^2}{a^2},\quad \cos^2 r_{,}=\frac{y''^2}{b^2},\quad \cos^2 r_{,,}=\frac{z''^2}{c^2};$$

we therefore have the known equations for the coordinates of the point of intersection of three confocal surfaces

$$x''^2=\frac{a^2\mu^2\nu^2}{(a^2-b^2)(a^2-c^2)},\quad y''^2=\frac{b^2\mu''^2\nu''^2}{(b^2-a^2)(b^2-c^2)},\quad z''^2=\frac{c^2\mu'''^2\nu'''^2}{(a^2-c^2)(b^2-c^2)}.$$

Now, if we suppose the point $(x'y'z')$ fixed in space and the surface ϖ given, it is manifest that this latter surface is the locus of the centre of the original ellipsoid; it is easy to see that this amounts to saying, "*If we have given in position the three points in which any three conjugate diameters of an ellipsoid terminate, together with its semi-major axis, the locus of the centre will be a surface of the second order.*" If two of its principal semi-axes be given, *i.e.* (suppose) the surfaces ϖ and ϖ', *the locus of the centre will evidently be the line of curvature made by their intersection.* These very beautiful theorems were first demonstrated by Prof. Chasles. If, as in the first case, we suppose the point $(x'y'z')$ fixed in space, the surface ϖ given and the ordinate (x') constant, *the locus of the centre will be a line of constant curvature upon the surface* ϖ, the constant measure of curvature being $\frac{x'^4}{a^2\mu^2\nu^2}$. In this case the bifocal chords of the auxiliary system $(\varpi, \varpi', \varpi'')$ will make a constant angle with the major axis of the constructed ellipsoid. This theorem is, so far as I am aware, stated for the first time.

In general, since, if P be the perpendicular from the centre of a surface of the second order upon the tangent

plane, and (a) the semi-major axis of the surface, we have

$$\cos r = \frac{P}{a},$$

where (r) denotes the angle which the bifocal chord drawn from the point of contact of the tangent plane makes with the normal to the surface at that point; it follows, that along the line of constant curvature, *the bifocal chord makes a constant angle with the normal to the surface.*

DEMONSTRATION OF A THEOREM OF JACOBI, RELATIVE TO FUNCTIONAL DETERMINANTS.

By Professor W. F. Donkin.

The following proposition is of fundamental importance in Jacobi's general theory of multipliers of a system of differential equations.

Let $u_1, u_2, \ldots u_n$ be n functions of the n variables $x_1, x_2, \ldots x_n$; and let Δ be the determinant (of n^2 terms) formed with the differential coefficients

$$\frac{du_1}{dx_1}, \frac{du_1}{dx_2}, \ldots \frac{du_1}{dx_n}; \quad \frac{du_2}{dx_1}, \ldots \frac{du_2}{dx_n}; \text{ \&c.}$$

If we put

$$\Delta = \frac{du_1}{dx_1}\nabla_1 + \frac{du_1}{dx_2}\nabla_2 + \ldots + \frac{du_1}{dx_n}\nabla_n,$$

$\nabla_1, \nabla_2, \ldots \nabla_n$ will be the n minor determinants obtained by leaving out the first horizontal row in Δ, and then, successively, each vertical row in the remaining terms. The proposition in question is expressed by the equation

$$\frac{d\nabla_1}{dx_1} + \frac{d\nabla_2}{dx_2} + \ldots + \frac{d\nabla_n}{dx_n} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (\text{I.})$$

(It will be observed that u_1 does not appear in this equation at all; it has been introduced merely for the sake of symmetry and clearness.)

This theorem may be very simply demonstrated by the help of the following property of determinants.

If the first term in each vertical (or in each horizontal) row be equal to the sum of the remaining terms in that row,

the determinant vanishes.* For it is plain that the determinant (of n^2 terms)

$$\begin{vmatrix} a_2 + a_3 + \ldots + a_n, & b_2 + b_3 + \ldots + b_n, \ldots\ldots & k_2 + k_3 + \ldots + k_n \\ a_2 \quad , & b_2 \quad , \ldots\ldots & k_2 \\ a_3 \quad , & b_3 \quad , \ldots\ldots & k_3 \\ \vdots \quad \vdots & \vdots \quad \vdots \ldots\ldots & \vdots \\ a_n \quad , & b_n \quad , \ldots\ldots & k_n \end{vmatrix}$$

is the product of two determinants, namely,

$$\begin{vmatrix} 0, & 1, & 1, \ldots 1 \\ 0, & 1, & 0, \ldots 0 \\ 0, & 0, & 1, \ldots 0 \\ \vdots & \vdots & \vdots \ldots \vdots \\ 0, & 0, & 0, \ldots 1 \end{vmatrix} \begin{vmatrix} a_1, & b_1, \ldots k_1, \\ a_2, & b_2, \ldots k_2, \\ a_3, & b_3, \ldots k_3, \\ \vdots & \vdots & \ldots \vdots \\ a_n, & b_n, \ldots k_n \end{vmatrix}$$

of which the first vanishes, since its first vertical row is composed wholly of zeros.

Returning now to the determinant Δ, it is evident that it may be represented in the form

$$D.u_1 u_2 \ldots u_n,$$

where D is an operation performed upon the product $u_1 u_2 \ldots u_n$, and defined by the equation

$$D = \begin{vmatrix} \frac{d_1}{dx_1}, & \frac{d_1}{dx_2}, \ldots. & \frac{d_1}{dx_n} \\ \frac{d_2}{dx_1}, & \frac{d_2}{dx_2}, \ldots. & \frac{d_2}{dx_n} \\ \vdots & \vdots & \vdots \\ \frac{d_n}{dx_1}, & \frac{d_n}{dx_2}, \ldots. & \frac{d_n}{dx_n} \end{vmatrix}$$

in which each symbol of differentiation is operative only upon that one of the functions u_1, u_2, &c., which has an index the same as that of d.

But if in the above expression for D we leave out the index 1 of d in each term of the first horizontal row, so that the symbols of differentiation in that one row may be unrestricted in their effect, and call the operation represented

* This might be assumed without a special demonstration, since it is obvious, from first principles, that if one and the same linear (homogeneous) function of the terms in each row vanish, the determinant vanishes.

by the determinant thus modified E, then it is also plain that

$$E.u_2u_3 \dots u_n,$$

will represent precisely the expression on the left of the equation (I.)

Now it follows from the form of the subject of operation $(u_2u_3 \dots u_n)$ that we may substitute for the unrestricted symbol $\frac{d}{dx_i}$, the expression

$$\frac{d_2}{dx_i} + \frac{d_3}{dx_i} + \dots + \frac{d_n}{dx_i}.$$

But by this substitution the determinant E is transformed into one in which the first term in each vertical row is the sum of the remaining terms in that row. Consequently E vanishes identically, and the equation (I) is established.

Oxford, Feb. 27, 1854.

ON A THEOREM OF M. LEJEUNE DIRICHLET'S.

By Arthur Cayley.

The following formula,

$$\Sigma q^{ax^2+2bxy+cy^2} + \Sigma q^{a'x^2+2b'xy+c'y^2} = 2\Sigma\delta^{\frac{n-1}{2}}\varepsilon^{\frac{n^2-1}{8}}\left(\frac{n}{P}\right)q^{nw'} \dots (3),$$

is given in Lejeune Dirichlet's well-known memoir "Recherches sur diverses applications, &c. (*Crelle*, tom. XXI. p. 8). The notation is as follows:—On the left-hand side (a, b, c), (a', b', c') ... are a system of properly primitive forms to the *negative* determinant D (*i.e.* a system of positive forms); x, y are positive or negative integers including zero, such that in the sum $\Sigma q^{ax^2+2bxy+cy^2}$, $ax^2 + 2bxy + cy^2$ is prime to $2D$, and similarly in the other sums; q is indeterminate and the summations extend to the values first mentioned, of x and y. On the right-hand side we have to consider the form of D, viz. we have $D = PS^2$ or else $D = 2PS^2$, where S^2 is the greatest square factor in D and where P is odd: this obviously defines P, and the values of δ, ε, which are always ± 1 (or, as I prefer to express it, are always $\pm$) are given as follows, viz.

$$D = PS^2, \quad P \equiv 1 \pmod 4, \quad \delta, \varepsilon = + +$$
$$D = PS^2, \quad P \equiv 3 \pmod 4, \quad \delta, \varepsilon = - +$$
$$D = 2PS^2, \quad P \equiv 1 \pmod 4, \quad \delta, \varepsilon = + -$$
$$D = 2PS^2, \quad P \equiv 3 \pmod 4, \quad \delta, \varepsilon = - -$$

n, n' are any positive numbers prime to $2D$, $\left(\frac{n}{P}\right)$ is Legendre's symbol as generalized by Jacobi, viz. in general if p be a positive or negative prime not a factor of n, then $\left(\frac{n}{p}\right) = +$ or $-$ according as n is or is not a quadratic residue of p (or, what is the same thing, p being positive, $\left(\frac{n}{\pm p}\right) \equiv n^{\frac{p-1}{2}} \pmod p$), and for $P = pp'p''\ldots$

$$\left(\frac{n}{P}\right) = \left(\frac{n}{p}\right)\left(\frac{n}{p'}\right)\left(\frac{n}{p''}\right),$$

and the summation extends to all the values of n, n' of the form above mentioned. In the particular case $D = -1$, it is necessary that the second side should be doubled. The method of reducing the equation is indicated in the memoir. The following are a few particular cases.

$$D = -1, \qquad \Sigma q^{x^2+y^2} = 4\Sigma(-)^{\frac{n-1}{2}} q^{nn'},$$

or
$$(1 + 2q^4 + 2q^{16} + 2q^{36} + \ldots)(q + q^9 + q^{25} + q^{49} + \ldots)$$
$$= \frac{q}{1-q^2} - \frac{q^3}{1-q^6} + \frac{q^5}{1-q^{10}} - \frac{q^7}{1-q^{14}} + \ldots$$

$$D = -2, \qquad \Sigma q^{x^2+2y^2} = 2\Sigma(-)^{\frac{n-1}{2} + \frac{n^2-1}{8}} q^{nn'},$$

or
$$(1 + 2q^2 + 2q^8 + 2q^{18}\ldots)(q + q^9 + q^{25} + q^{49} + \ldots)$$
$$= \frac{q}{1-q^2} + \frac{q^3}{1-q^6} - \frac{q^5}{1-q^{10}} - \frac{q^7}{1-q^{14}} + \&c.$$

an example given in the memoir.

$$D = -3, \qquad \Sigma q^{x^2+3y^2} = 2\Sigma(\tfrac{1}{3}n)\, q^{nn'},$$

or
$$(q^1 + q^{25} + q^{49} + q^{121} + q^{169}\ldots)(1 + 2q^{12} + 2q^{48} + 2q^{108}\ldots)$$
$$+ 2(q^3 + q^{27} + q^{75} + q^{147} + \ldots)(q^4 + q^{16} + q^{64} + q^{100}\ldots)$$
$$= \frac{q + q^5}{1-q^6} - \frac{q^5 + q^{25}}{1-q^{30}} + \frac{q^7 + q^{35}}{1-q^{42}} - \frac{q^{11} + q^{55}}{1-q^{66}} + \ldots$$

I am not aware that the above theorem is quoted or referred to in any subsequent memoir on Elliptic Functions, or on the class of series to which it relates, and the theorem is so distinct in its origin and form from all other theorems relating to the same class of series, and, independently of the researches in which it originates, so remarkable as a result, that I have thought it desirable to give a detached statement of it in this paper.

2, *Stone Buildings, Lincoln's Inn,*
March 8, 1854.

MATHEMATICAL NOTES.

To the Editor of the Mathematical Journal.

In a note appended to a paper which appears in the recently published number of the *Journal* (No. xxxiv. p. 76) the late Mr. Weddle has kindly pointed out slight errors in the form in which two geometrical theorems are given by me in a paper published in the seventh volume of the *Journal.* The modifications in their enunciations necessary to rectify these errors, which Mr. Weddle has suggested, are at once seen to be proper; and I can only attribute to the haste with which these two special applications of a general theorem were written, my oversight of the correct method of deducing them. My object in addressing this note to you is that of briefly showing how these two theorems, as corrected by Mr. Weddle, are to be deduced agreeably to the method employed in my paper alluded to.

I conceive a surface S of the second order to be cut in two small conics by the planes L, L', and a cone to be described passing through the two conics. By the general theorem on which the method of the paper is founded, the rectangle of segments of a secant of the cone parallel to a circular section, drawn from any point in S, will bear a constant ratio to the rectangle of perpendiculars let fall from the same point on the planes L, L'. Then, supposing the planes L, L' moving parallel to themselves to become tangent planes, and the cone to degenerate into one of infinitesimal aperture, the first of the two corrected theorems will be true; the fixed plane to which the line drawn to meet the chord of contact is parallel, being one parallel to the ultimate direction of the circular section of the cone. Again, if a second surface S', likewise of the second order, be conceived to touch the

planes L, L' in the same two points as S, the rectangle of the perpendiculars let fall from any point in S upon the planes will bear a constant ratio to the segments of any chord (or secant) of S' drawn from the same point parallel to a fixed line; and by compounding ratios it will follow that the rectangle of the segments of a chord (or secant) of S' drawn from any point in S parallel to a fixed line, will bear a constant ratio to the square of a line drawn from the same point to the chord of contact parallel to a certain fixed plane.

Writing on a geometrical subject, I am induced to send you the following remarks on two geometrical interpretations of which the equation to the circle is capable in its general form. They occurred to me some time back, and perhaps you might think them not out of place among your Mathematical Notes, as the first of the two ought, I conceive, to find a place in all elementary treatises on analytic geometry.

The particular values of the coefficients of the terms x^2, xy, y^2 in the general equation of the second order when it represents a circle express two well-known *metrical* properties of that curve, the opposite sides of an inscribed tetragon being chosen as axes of coordinates. To show the former of these it is necessary to premise the following:

LEMMA. If the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \ldots\ldots\ldots (a)$$

represent two right lines, and ϕ be the angle at which the axes of coordinates are inclined, the rectangle of perpendiculars let fall from a point $x'y'$ on the two right lines will be given by the formula

$$\frac{(Ax'^2 + Bx'y' + Cy'^2 + Dx' + Ey' + F)\sin^2\phi}{\{(A-C)^2 - 2B(A+C)\cos\phi + B^2 + 4AC\cos^2\phi\}^{\frac{1}{2}}} \ldots (b).$$

For if $\alpha x + \beta y + \gamma$ and $\alpha' x + \beta' y + \gamma'$ represent the two factors of the first degree into which the left-hand side of the equation (a) may be resolved, the perpendiculars will be respectively,

$$\frac{(\alpha x' + \beta y' + \gamma)\sin\phi}{(\alpha^2 - 2\alpha\beta\cos\phi + \beta^2)^{\frac{1}{2}}} \text{ and } \frac{(\alpha' x' + \beta' y' + \gamma')\sin\phi}{(\alpha'^2 - 2\alpha'\beta'\cos\phi + \beta'^2)^{\frac{1}{2}}}.$$

The product of the denominators of these fractions is

$$[\alpha^2\alpha'^2 + \alpha^2\beta'^2 + \alpha'^2\beta^2 + \beta^2\beta'^2 - 2\{\alpha\beta(\alpha'^2 + \beta'^2) + \alpha'\beta'(\alpha^2 + \beta^2)\}\cos\phi + 4\alpha\alpha'\beta\beta'\cos^2\phi]^{\frac{1}{2}}.$$

But from comparing the product of their numerators with the expression

$$(Ax^2 + Bxy + Cy^2 + Dx + Ey + F)\sin^2\phi,$$

it appears that

$$\alpha\alpha' = A, \quad \beta\beta' = C; \quad \alpha^2\beta'^2 + \alpha'^2\beta^2 = B^2 - 2AC;$$

$$\alpha\beta(\alpha'^2 + \beta'^2) + \alpha'\beta'(\alpha^2 + \beta^2) = (\alpha\alpha' + \beta\beta')(\alpha\beta' + \alpha'\beta) = (A + C)B,$$

whence the product of the two fractions may be easily identified with the formula (*b*).

If in the equation of the two right lines we suppose $A = C = 1$, (*i.e.* that the two right lines are subcontrary one to the other with respect to the axes,) the expression for the rectangle of perpendiculars from $x'y'$ becomes

$$\frac{(x'^2 + Bx'y' + y'^2 + Dx' + Ey' + F)\sin^2\phi}{B - 2\cos\phi} \quad \ldots\ldots (c).$$

Suppose now the axes of coordinates to be the opposite sides of a tetragon inscribed in a circle, and B to be a certain constant quantity; the equation of the circle may easily be thrown into the form

$$\frac{(x^2 + Bxy + y^2 + Dx + Ey + F)\sin^2\phi}{B - 2\cos\phi} = xy\sin^2\phi \ldots (d).$$

Let B now be assumed equal to either of the expressions

$$\frac{DE \pm \{(D^2 - 4F)(E^2 - 4F)\}^{\frac{1}{2}}}{2F},$$

and the equation (*d*) will at once be seen to express the equality of the rectangles of the three pairs of perpendiculars let fall from any point in a circle on the three pairs of opposite sides of an inscribed tetragon.

In the same case, supposing $x'y'$ to be the point of intersection of a pair of opposite sides of the tetragon, it is easily shown that $Dx' + Ey' = -2F$. Let l be the length of the line drawn from the origin to $x'y'$, and t, t' the lengths of tangents drawn from the same points; then

$$\pm t'^2 = x'^2 + 2\cos\phi x'y' + y'^2 + Dx' + Ey' + F$$

$$= x'^2 + 2\cos\phi x'y' + y'^2 - F = l^2 - t^2,$$

or

$$l^2 = t^2 \pm t'^2,$$

the upper or lower sign being used according as $x'y'$ is without or within the circle (the origin being without), in which

latter case t' must be considered as half the chord drawn through $x'y'$ so as to be bisected at that point. This is another well-known metrical property of the circle.

By pursuing a very similar method of investigation, this theorem may be slightly generalized thus: Suppose a circle and any central conic to circumscribe the same tetragon; the sum of the squares of tangents drawn to the circle from the point of intersection of a pair of opposite sides of the tetragon, and from the centre of the conic is equal to the difference between the square of the line joining those points and the sum of squares of the semi-diameters of the conic which are parallel to that pair of opposite sides of the tetragon.

JOHN WALKER.

Dollymount, near Dublin,
Feb. 27, 1854.

SOME FORMULÆ IN FINITE DIFFERENCES.

Since $\log(1+\Delta)=\dfrac{d}{dx}$; therefore,

$$\psi\{x+\log(1+\Delta)\}\,y^n=\psi\left(x+\frac{d}{dy}\right)y^n,$$

and clearly, when $y=0$, the only term in the development of $\psi\left(x+\dfrac{d}{dy}\right)y^n$ which does not vanish, is

$$\frac{1}{1.2\ldots n}\psi^{(n)}(x)\,\frac{d^n}{dy^n}(y^n) \text{ or } \psi^{(n)}(x);$$

therefore $$\frac{d^n}{dx^n}\psi(x)=\psi\{x+\log(1+\Delta)\}0^n.$$

Putting $x=0$ and using Maclaurin's series, we obtain Herschel's theorem. Again, let

$$\psi(x)=B_0+B_1x+\ldots+B_rx^r+\ldots;$$

therefore $$\psi\left(\frac{d}{dx}\right)\phi(x)=B_0\phi(x)+\ldots+B_r\phi\{x+\log(1+\Delta)\}0^r+\ldots;$$

therefore $$\psi\left(\frac{d}{dx}\right)\phi(x)=\phi\{x+\log(1+\Delta)\}\,\psi(0).$$

Whence it easily follows that

$$F(\Delta)\,\psi(0)=\psi\left(\frac{d}{dx}\right)F(\varepsilon^x-1) \text{ when } x=0.$$

Let $$F(\varepsilon^x-1)=C_0+C_1x+\ldots+C_rx^r+\ldots;$$

therefore $F(\Delta)\,\psi(0)$ = value of

$$\left\{\psi(0)+\ldots+\psi^{(p)}(0)\,\frac{d^p}{dx^p}+\ldots\right\}(C_0+\ldots+C_px^p\ldots) \text{ when } x=0,$$

$$=\psi(0)\,C_0+\psi'(0)\,C_1+\ldots+\psi^{(p)}\,C_p+\ldots.$$

The following results are consequences of this formula.

$$S_n=\psi(0)+\psi(1)+\ldots+\psi(n-1)=\frac{(1+\Delta)^n-1}{\Delta}\,\psi(0)$$

$$=\psi\left(\frac{d}{dx}\right)\left(\frac{\varepsilon^{nx}-1}{\varepsilon^x-1}\right) \text{ when } n=0,$$

$$=n\psi(0)+\psi'(0)\,S_1+\ldots+\psi^{(p)}(0)\,\frac{S_p}{\lfloor p}+\ldots,$$

where $$S_p=1^p+\ldots+(n-1)^p.$$

Also $$\frac{\log(1+\Delta)}{\Delta}\,\psi(0)=\psi(0)-\frac{\psi'(0)}{2}+\ldots+(-1)^{n+1}\,B_{2n-1}\,\frac{\psi^{2n}(0)}{\lfloor 2n}+\ldots,$$

B_{2n-1} being one of Bernoulli's numbers.

ARTHUR COHEN.

THE following proof of Legendre's theorem (here very briefly given) may be worth attention, if it be not already known. The notation is as usual; accented letters being the angles of the plane triangle. By common substitution for $\sin A$, $\cos A$, &c.

$$\sin(A-B)=\frac{2\sqrt{(\sin s.\sin s-a.\sin s-b.\sin s-c)}}{\sin a\,\sin b\,\sin^2 c}(\cos b-\cos a)(1+\cos c),$$

$$\sin(A'-B')=\frac{2\sqrt{(s.s-a.s-b.s-c)}}{abc^2}\cdot\frac{a^2-b^2}{2}(1+1).$$

Hence, using $\frac{\sin x}{x}=1-\frac{x^2}{6}+\ldots$ &c., and expanding

$$\sin(A-B):\sin(A'-B'),$$

we find for this ratio the product of nine series of the form 1 + term of second order + term of fourth order + &c., the nine terms of the second order being

$$-\frac{s^2}{12},-\frac{(s-a)^2}{12},-\frac{(s-b)^2}{12},-\frac{(s-c)^2}{12},+\frac{a^2}{6},+\frac{b^2}{6},+\frac{c^2}{3},-\frac{a^2+b^2}{12},-\frac{c^2}{4},$$

the sum of which vanishes. Hence, so far as terms of the second order, the ratio of the sines, and that of the angles, is a ratio of equality, or $A - A' = B - B' =$ (by similar reasoning) $C - C'$; whence Legendre's theorem immediately follows.

A. DE MORGAN.

LAGRANGE, in a paper on spherical triangles (*Jour. Ec. Polytechn.*, vol. II., translated in Leybourn's *Repository*, vol. I.) notes that the dependence of the area on the spherical excess, first announced by Albert Girard, in 1629, was not demonstrated by him, but was first demonstrated by Cavalieri, in his *Directorium, &c.*, published in 1632. Lagrange therefore considers that the theorem belongs rather to Cavalieri than to Albert Girard. This transfer, however, cannot be made. It is a matter of fact that Girard discovered the theorem, and Cavalieri the demonstration; it is also a matter of fact, that for one who can discover, there are hundreds who can demonstrate, when they know what is to be demonstrated. At the same time, looking at dates and countries, it is most probable that Cavalieri was an independent discoverer.

As the *Invention Nouvelle*, &c. of Albert Girard is the rarest of all the works in which mathematical discovery has been announced, a short statement of what is there found may be acceptable. Girard considers the surface of the sphere as composed of 720 equal *superficial degrees;* and then announces and exemplifies the theorem, or *science*, "incogneuë jusques à present, si ce n'est devant le deluge," that every spherical polygon has as many superficial degrees as the sum of all the angles contains degrees more than the sum of all the angles of a corresponding rectilinear figure. This he first proves, easily of course, of a semilune, or *fibulle*, as he calls it. He then proceeds to the case of a right-angled triangle having each side less than a quadrant; and this case he does not attempt to prove except *en conclusion probable*. The triangle being ABC, right-angled at C, he shows that if a small circle be drawn having A for its pole, and cutting the circles AB, AC in M and N, so that the area AMN may be equal to the semilune which has the same spherical excess as ABC, then M falls within AB, and N on AC produced, so that MN cuts BC, say in P. If he could show MPB, CPN to be equal, the theorem would be proved; and here he stops, calling the theorem *manifeste*

et probable. In the steps which follow this assumption there is nothing to notice. It should be remarked, that what Girard does prove is very much more difficult than the general theorem. Had he thought of the manner in which three circles divide the sphere into eight triangles, the demonstration given by Cavalieri would not have been concealed for one minute from the geometer who could demonstrate that MN cuts BC.

A. DE MORGAN.

To eliminate x, y, λ from

$$\begin{aligned} ax + by + \lambda(\alpha x + \beta y) &= 0, \\ a'x + b'y + \lambda(\alpha' x + \beta' y) &= 0, \\ a''x + b''y + \lambda(\alpha'' x + \beta'' y) &= 0. \end{aligned}$$

Assume $1 + \lambda\mu = 0$, then ξ, η being arbitrary,

$$\xi x + \eta y + \lambda(\mu\xi x + \mu\eta y) = 0.$$

And eliminating x, y, λx, λy from the four equations, therefore

$$\begin{vmatrix} \xi & \eta & \mu\xi & \mu\eta \\ a & b & \alpha & \beta \\ a' & b' & \alpha' & \beta' \\ a'' & b'' & \alpha'' & \beta'' \end{vmatrix} = 0,$$

an equation which may be written

$$A\xi + B\eta + C\mu\xi + D\mu\eta = 0;$$

and the equation being true, independently of the values of ξ, η, we have

$$\begin{aligned} A + C\mu &= 0, \\ B + D\mu &= 0. \end{aligned}$$

Or, eliminating μ,

$$AD - BC = 0,$$

which is the result of the elimination of x, y, λ from the given equations, or, what is the same thing, the result of the elimination of x, y from the equations

$$\frac{ax + by}{\alpha x + \beta y} = \frac{a'x + b'y}{\alpha' x + \beta' y} = \frac{a''x + b''y}{\alpha'' x + \beta'' y}.$$

A. C.

To the Editors of the Cambridge and Dublin Mathematical Journal.

GENTLEMEN,

I request the insertion of the following statement in the next number of your *Journal.*

Having read a paper by Mr. Matthew Collins "On Clairaut's Theorem," which appeared in the February number of the *Journal,* I think it right to state, that the substance of that communication has been taken, without acknowledgment, from a series of Lectures delivered by the late Professor MacCullagh in the University of Dublin, in Hilary and Michaelmas Terms, 1846.

I attended the Lectures alluded to, in company with Mr. Collins amongst others; and a memoir containing an account of them has been prepared by me for the Royal Irish Academy, and was published some months since in its *Transactions.*

In the course of last year Mr. Collins published a pamphlet absolutely identical with the paper which appeared in the *Journal.* He sent several copies of this pamphlet to distinguished members of the University of Dublin. I have now before me one of these copies, on the title-page of which Mr. Collins has *written* the following acknowledgment—"Extracted from manuscript notes taken at MacCullagh's Lectures."

I have the honor to be, Gentlemen,

Your obedient Servant,

GEORGE J. ALLMAN, LL.D.,

Professor of Mathematics, Queen's College, Galway.

April 10, 1854.

SUR LA THEORIE DES FONCTIONS HOMOGENES À DEUX INDETERMINEES.

Par M. HERMITE.

MES premières recherches sur la théorie des formes à deux indéterminées, ont pour objet la démonstration de cette proposition arithmétique élémentaire, *que les formes à coefficients entiers et en nombre infini, qui ont les mêmes invariants, ne donnent qu'un nombre essentiellement limité de classes distinctes.*

Une notion générale sur les invariants s'est offerte dans ces recherches, amenée par une considération purement arithmétique, la réduction des simples formes quadratiques définies, et l'application très facile que j'ai pu faire pour les formes cubiques et biquadratiques, m'a donné leurs invariants obtenus déja par M. Cayley, en suivant une toute autre voie. Mais à partir du cinquième degré l'application de cette méthode devenait si pénible, que j'ai du renoncer à l'espoir d'en tirer *explicitement* les expressions de leurs invariants, et à plus forte raison celle des invariants des formes des degrés plus élevés. Ramené demièrement à ces questions, j'ai été conduit à les envisager sous un point de vue nouveau, et j'ai pu enfin aborder les formes du cinquième degré, qui n'avaient pu être traitées par ma première méthode. Les circonstances singulières, que j'ai rencontrées dans cette recherche, me semblent ajouter encore à l'interet de la grande théorie que MM. Cayley et Sylvester ont déja enrichie de tant de découvertes. Mais j'ai eu surtout en vue la théorie arithmétique, dont j'ai ainsi trouvé les véritables éléments, comme l'on verra par la suite de mes recherches : dès à présent néanmoins on pourra reconnâitre que la théorie des formes binaires, dans toute sa généralité, est étroitement liée à la *composition des classes quadratiques*, résultat singulier et qui ouvrera des nouvelles perspectives dans l'étude des propriétés les plus cachées des nombres. La loi de réciprocité dont M. Sylvester a bien voulu déja annoncer la découverte, étant le point de départ de mon analyse, je dira d'abord en peu de mots en quoi elle consiste.

Section I.—Loi de Réciprocité.

Elle est contenue dans le théoréme : *A tout covariant d'une forme de degré m, et qui par rapport aux coefficients de cette forme est du degré p, correspond un covariant du degré m par rapport aux coefficients, d'une forme du degré p.*

Soit,

$$f(x, y) = a(x + \alpha y)(x + \alpha' y)\ldots(x + \alpha^{(m-1)} y) = a \text{ Norme } (x + \alpha y),$$

une forme du degré m décomposée en facteurs linéaires, et $\phi(x, y)$, un covariant de cette forme du degré p quant aux coefficients, et d'un degré quelconque en x et y.

Si nous faisons

$$F = a^p \text{ Norme } (X + \alpha Y + \alpha^2 Z + \ldots + \alpha^p T),$$

les coefficients de F, seront des fonctions entières de degré p,

des coefficients de f, et on démontrera facilement ces deux lemmes.

1°. *Toute fonction entière et du degré p des coefficients de f, s'exprime linéairement par ceux de la forme F.*

2°. *Les coefficients de F ne sauraient être liés par aucune relation du premier degré dont les coefficients seraient numériques,* c. à. d. *indépendents des coefficients de f. D'où résulte qu'une fonction du degré p de ces coefficients n'est absolument susceptible que d'une seule expression linéaire par ceux de F.*

Cela étant, voici comment du covariant $\phi(x, y)$ qui se rapporte à la forme f, du degré m, se déduit un covariant se rapportant à une forme du degré p.

Soit,

$$f(x, y) = ax^m + mbx^{m-1}y + \frac{m.m-1}{1.2}\, cx^{m-2}y^2 + \text{etc.}$$

de sorte que les constantes a, b, c.... soient ce que nous avons appelé les coefficients de f; nous leurs donnerons une désignation plus expressive, en les représentant de cette manière

$$a = (x_0^m), \quad b = (x_0^{m-1}y_0), \quad c = (x_0^{m-2}y_0^2), \ldots$$

ainsi l'expression de $f(x, y)$ deviendra par la suppression des parenthèses la puissance

$$(xx_0 + yy_0)^m.$$

Faisons de même

$$F = a^p \text{ Norme } (X + \alpha Y + \alpha^2 Z + \ldots + \alpha^p T)$$
$$= (XX_0 + YY_0 + \ldots + TT_0)^m,$$

en convenant après le developpement de la puissance d'écrire

$$p.\ ex.\ (X_0^m)\, X^m, \quad (X_0^{m-1}Y_0)\, X^{m-1}Y, \text{ etc.},$$

respectivement au lieu de

$$X_0^m X^m, \quad X_0^{m-1}Y_0, \quad X^{m-1}Y, \text{ etc.},$$

ce qui sera une désignation commode des coefficients de F. Cela posé, d'après le premier des lemmes ci-dessus, on pourra, et d'une manière seulement, exprimer linéairement les coefficients du covariant $\phi(x, y)$, par les quantités

$$(X_0^m), \ (X_0^{m-1}Y), \text{ etc.};$$

or il se présente cette conséquence remarquable,

Ayant exprimé $\phi(x, y)$ par les quantités

$$(X_0^m),\ (X_0^{m-1}Y),\ \text{etc.},$$

conçevons que l'on supprime les parenthèses, on arrivera par là à une fonction du m^e degré par rapport aux quantités

$$X_0,\ Y_0,\ Z_0,\ \ldots\ T_0.$$

Or cette fonction sera un covariant de la forme suivante de degré p,

$$X_0x^p + pY_0x^{p-2} + \frac{p.p-1}{1.2}Z_0x^{p-2}y^2 + \ldots + T_0y^p.$$

Rien d'ailleurs ne vient ici changer le degré des indéterminées x et y, dans cette métamorphose que subit la fonction $\phi(x, y)$, ainsi ce sont des covariants de même degré par rapport aux indéterminées, qui se trouvent liés l'un à l'autre par la loi de réciprocité. Mais il y a une seconde manière de passer ainsi d'un covariant se rapportant à une forme d'un certain degré, à un covariant se rapportant à une forme d'un autre degré. L'analyse précédente conduit en effet, et très aisément, a ce second théorème :

Etant donné un covariant quelconque du m^e degré par rapport aux coefficients de la forme

$$X_0x^p + pY_0x^{p-1}y + \frac{p.p-1}{1.2}Z_0x^{p-2}y^2 + \ldots + T_0y^p.$$

Si l'on transforme en symboles, dans l'expression de ce covariant, les quantités X_0^m, $X_0^{m-1}Y$, etc., ... en les remplacant respectivement par (X_0^m), $(X_0^{m-1}Y)$, etc., coefficients de la forme F, ce covariant se transformera en un autre se rapportant à la forme $f(x, y)$, et du degré p, relativement aux coefficients de cette forme.

Section II.—Conséquences de la Loi de Réciprocité.

Nous considérons en premier lieu les invariants qui sont un cas particulier des covariants, lorsqu'on suppose leur degré nul par rapport aux indéterminées x et y. La connaissance complète que nous avons des invariants des formes du second, troisième, et quatrième degré, nous donnera alors immediatement pour des formes de degré quelconque, les invariants qui sont du second, troisième, et quatrième degré par rapport aux coefficients de ces formes. Ainsi les formes quadratiques

$$f = ax^2 + 2bxy + cy^2 = a(x + \alpha y)(x + \alpha' y),$$

ont pour expression générale de leurs invariants, la fonction de degré 2μ,

$$\Delta = (b^2 - ac)^\mu = a^{2\mu}(\alpha - \alpha')^{2\mu},$$

donc toutes les formes de degré 2μ possèdent un invariant quadratique, que nous allons calculer. Soit pour cela

$$F = a^{2\mu}(X + \alpha X' + \alpha^2 X'' + \ldots + \alpha^{2\mu} X^{(2\mu)})(X + \alpha' X' + \alpha'^2 X'' + \ldots + \alpha'^{2\mu} X^{(2\mu)});$$

en représentant symboliquement cette forme, suivant notre convention par

$$F = (XX_0 + X'X_0' + \ldots + X^{(2\mu)}X_0^{(2\mu)})^2,$$

on trouvera bien aisément

$$(X_0^{(i)})^2 = a^{2\mu}\alpha^i\alpha'^i,$$

$$2(X_0^{(i)}X_0^{(j)}) = a^{2\mu}(\alpha^i\alpha'^j + \alpha^j\alpha'^i).$$

Maintenant il viendra par le developpement de la puissance

$$\Delta = a^{2\mu}(\alpha - \alpha')^{2\mu}$$

$$= a^{2\mu}[(\alpha^{2\mu} + \alpha'^{2\mu}) - \mu_1(\alpha^{2\mu-1}\alpha' + \alpha'^{2\mu-1}\alpha) + \mu_2(\alpha^{2\mu-2}\alpha'^2 + \alpha'^{2\mu-2}\alpha^2) - \text{etc.}],$$

en rapprochant les termes équidistants des extrèmes, et nommant pour abreger μ_1, μ_2,... les coefficients binomiaux. Cela fait on peut immediatement introduire les coefficients de F, et il viendra

$$\Delta = 2(X_0X_0^{(2\mu)}) - 2\mu_1(X_0'X_0^{(2\mu-1)}) + 2\mu_2(X_0''X_0^{(2\mu-2)}) - \text{etc.},$$

le dernier terme qui seul ne contient pas en evidence le facteur 2, étant

$$(-1)^\mu \mu^\mu (X_0^{(\mu)}X_0^{(\mu)}).$$

Or tel est l'invariant quadratique de la forme

$$X_0x^{2\mu} + \mu_1X_0'x^{2\mu-1}y + \mu_2X_0''x^{2\mu-2}y^2 + \ldots + X_0^{(2\mu)}y^{2\mu},$$

dont M. Cayley le premier a fait la découverte.

Les formes cubiques

$$f = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

ont pour invariants la fonction de degré 4μ,

$$\Delta = (a^2d^2 - 3b^2c^2 - 6abcd + 4b^3d + 4ac^3)^\mu,$$

donc toutes les fonctions du degré 4μ et celles-là seules, possèdent un invariant du troisième degré. Le théorème conduirait si l'on ne le connaissait pas déja, à l'invariant cubique des formes biquadratiques

$$f = ax^4 + 4bx^3y + 6cx^2y^2 + 4dxy^3 + ey^4.$$

Nous avons d'ailleurs, l'invariant quadratique

$$\Delta = ae - 4bd + 3c^2,$$

et si nous posons

$$\Delta' = ace + 2bcd - ad^2 - c^3 - b^2e,$$

l'expression générale des invariants des formes biquadratiques, sera la fonction de degré $2m + 3n$,

$$\Delta^m\Delta'^n,$$

comme l'a démontré M. Sylvester. Donc toutes les formes de degré $\mu = 2m + 3n$, possédent des invariants du quatrième degré en nombre égal à celui des solutions entières et positives de cette équation $\mu = 2m + 3n$. C'est là encore un des beaux résultats obtenus par M. Cayley dans son mémoire sur les hyperdéterminants. Mais les conséquences de la loi de réciprocité, dont j'aurai besoin principalement dans la suite, se rapportant aux covariants, j'y arrive immediatement en omettant beaucoup de remarques auxquelles les résultats précédents donneraient lieu.

Considérant d'abord les formes quadratiques

$$f = ax^2 + 2bxy + cy^2,$$

nous avons cette expression générale de leurs covariants savoir,

$$\phi = (b^2 - ac)^\mu(ax^2 + 2bxy + cy^2)^\nu = a^{2\mu+\nu}(\alpha - \alpha')^{2\mu}(x + \alpha y)^\nu(\alpha x + \alpha' y)^\nu,$$

de degré $2\mu + \nu$ par rapport aux coefficients de f. Donc faisant

$$2\mu + \nu = m,$$

nous aurons autant de covariants du second degré par rapport aux formes de degré m, qu'il y a de solutions entières et positives de cette équation. D'ailleurs le nombre 2ν représente le degré de chacun de ces covariants en x et y. Dans le cas ou m est impair, et dans ce cas seulement, on peut faire $\nu = 1$, on est alors conduit à un covariant du second degré en x et y, dont nous allons donner l'expression générale a cause de son importance. A cet effet posons

$$m = 2\mu + 1, \quad \phi = Ax^2 + Bxy + Cy^2,$$

de sorte que

$$A = a^{2\mu+1}(\alpha - \alpha')^{2\mu}, \quad B = a^{2\mu+1}(\alpha - \alpha')^{2\mu}(\alpha + \alpha'),$$
$$C = a^{2\mu+1}(\alpha - \alpha')^{2\mu}\alpha\alpha',$$

il s'agira d'exprimer ces diverses quantités au moyen des coefficients de la forme

$$F = a^m(X + \alpha X' + \alpha^2 X'' + \ldots + \alpha^m X^{(m)})(X + \alpha' X' + \alpha'^2 X'' + \ldots + \alpha'^m X^{(m)}),$$

coefficients dont nous avons précédemment employé les valeurs savoir

$$2(X_0^{(i)}X_0^{(j)}) = a^m(\alpha^i\alpha'^j + \alpha^j\alpha'^i).$$

Or on trouve immediatement A et C, par le même calcul qui nous a donné l'invariant quadratique des formes de degré pair, savoir

$$A = 2(X_0^{(m-1)}X_0) - 2\mu_1(X_0^{(m-2)}X_0') + 2\mu_2(X_0^{(m-3)}X_0'') - \text{etc.},$$

$$C = 2(X_0^{(m)}X_0') - 2\mu_1(X_0^{(m-1)}X_0') + 2\mu_2(X_0^{(m-2)}X_0''') - \text{etc.},$$

quant a B, après quelques réductions très faciles on obtiendra

$$B = 2(X_0^{(m)}X_0) - 2(\mu_1 - 1)(X_0^{(m-1)}X_0') + 2(\mu_2 - \mu_1)(X_0^{(m-2)}X_0'') - \text{etc.},$$

la dernier terme étant

$$2(\mu_\mu - \mu_{\mu-1})(X_0^{(\mu+1)}X_0^{(\mu)}).$$

Pour les formes cubiques,

$$f = ax^3 + 3bx^2y + 3cxy^2 + dy^3,$$

nous avons le covariant quadratique

$$(b^2 - ac)x^2 + (bc - ad)xy + (c^2 - bd)y^2,$$

que donneraient les formules précédentes, mais en le multipliant par une puissance μ de l'invariant du 4e degré, on obtient un covariant du degré $4\mu + 2$ par rapport aux coefficients de f; donc toutes les formes de degré $4\mu + 2$ ont un covariant quadratique en x et y, et du troisième degré par rapport à leurs coefficients. Cette conclusion à laquelle il eut peut-être été difficile de parvenir par une autre voie, nous révéle ainsi l'existence d'un covariant quadratique pour toutes les formes dont la degré n'est pas un multiple de 4. Ces dernières comme nous pourrons l'établir plus tard, possédent elles-mêmes un covariant quadratique, du 5e degré par rapport à leurs coefficients, les seules formes biquadratiques exceptées. Les considerations dans lesquelles nous allons entrer, vont montrer la grande importance de ces covariants quadratiques.

Section III.—Des Formes Canoniques.

Soit $f(x, y)$ une forme pour laquelle on ait reconnu l'existence d'un covariant du second degré en x et y,

$$\phi(x, y) = Ax^2 + 2Bxy + Cy^2.$$

Posons $\Delta = B^2 - AC$, il existera comme on sait une infinité

de substitutions, au déterminant *un*, propres à faire disparaitre les coefficients des carrés des indéterminées, et à réduire ϕ à l'expression suivante

$$XY.\sqrt{\Delta}.$$

Soit

$$\left.\begin{aligned} x &= \alpha X + \beta Y \\ y &= \gamma X + \delta Y \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

l'une quelconque de ces substitutions, toutes les autres s'en déduiront comme on sait, en la faisant suivre de celle-ci

$$X = \omega\eta, \quad Y = -\frac{1}{\omega}\xi \ldots\ldots\ldots\ldots (2),$$

on ω est une quantité arbitraire. Cela posé, nous définirons comme forme canonique de $f(x, y)$, la transformée qui en résulte par la substitution (1). Cette forme canonique contiendra essentiellement dans les coefficients une quantité arbitraire qu'on mettra en évidence si l'on veut, en y faisant la substitution (2).

Mais posons d'abord

$$f(\alpha X + \beta Y,\ \gamma X + \delta Y) = F(X, Y),$$

nous aurons cette proposition fondamentale : *Toute fonction entière des coefficients de F, qui se reproduit identiquement dans la transformée obtenue par la substitution* (2), *est une fonction rationelle des coefficients de la proposée* $f(x, y)$, *le dénominateur de cette fonction étant une puissance de* Δ, *et la numérateur un invariant de f. En second lieu, toute fonction qui se reproduit au signe près, redonne si on la multiplie par* $\sqrt{\Delta}$, *le même expression que les précédentes.*

Voici donc la principe d'une nouvelle méthode pour la recherche des invariants, puisque tout invariant de la forme $f(x, y)$, s'exprime par une fonction semblable des coefficients de la transformée F, qui possédera evidemment la propriété mentionnée dans notre proposition. Nous allons en faire l'application aux formes du cinquième degré.

Section IV.

Recherche des Invariants des Formes du Cinquième Degré.

Nous représenterons la forme proposée par

$$f(x, y) = ax^5 + 5bx^4y + 10cx^3y^2 + 10c'x^2y^3 + 5b'xy^4 + a'y^5.$$

Le covariant quadratique, par

$$\phi = (ab' - 4bc' + 3c^2)\,x^2 + (aa' - 3bb' + 2cc')\,xy + (a'b - 4b'c + 3c'^2)\,y^2,$$

et enfin la transformée canonique par

$$F = AX^5 + 5BX^4Y + 10CX^3Y^2 + 10C'X^2Y^3 + 5B'XY^4 + A'Y^5.$$

Cela posé, puisque le covariant quadratique de F, se reduit par hypothése à l'expression $XY\sqrt{\Delta}$, nous aurons entre les coefficients de F, les relations suivantes,

$$AB' - 4BC + 3C^2 = 0, \quad AA' - 3BB' + 2CC' = \sqrt{\Delta},$$
$$A'B - 4B'C + 3C'^2 = 0,$$

et c'est sous ces conditions, qu'il nous faut obtenir l'expression la plus générale d'une fonction entière des coefficients de F, qui ne change pas en y faisant la substitution

$$X = \omega\eta, \quad Y = -\frac{1}{\omega}\xi.$$

Une analyse plus longue que difficile, et que je n'ai pas encore assez simplifiée pour l'exposer ici, m'a donné les propositions suivantes.

1°. *Toute fonction entière des coefficients A, B, C, etc., qui ne change pas quand on transforme F, par la substitution*

$$X = \omega\eta, \quad Y = -\frac{1}{\omega}\xi,$$

est nécessairement de degré pair.

2°. *Désignant par μ ce degré, si l'on a $\mu \equiv 0 \pmod 4$, l'expression la plus générale d'une telle fonction sera*

$$I = \Theta(AA', BB', CC'),$$

Θ étant homogène et de degré $\frac{1}{2}\mu$.

3°. *Si l'on a $\mu \equiv 2 \pmod 4$, l'expression la plus générale sera*

$$I = (ACB'^2 - A'C'B^2)\,\Theta_1(AA', BB', CC'),$$

Θ_1 étant une fonction homogène de degré $\frac{1}{2}\mu - 2$.

De cette dernière proposition découle immédiatement l'existence d'invariants de degré impairement pair, pour les formes du cinquième degré; en effet si nous considerons l'expression $ACB'^2 - A'C'B^2$, qui change de signe par la substitution $X = \omega\eta$, $Y = -\frac{1}{\omega}\xi$, il résulte du théorème établi dans la 3e §, qu'en la multipliant par $\sqrt{\Delta}$, le produit sera nécessairement de la forme $\frac{I}{\Delta^k}$, I étant un invariant de

$f(x, y)$, on a donc

$$I = \Delta^k\sqrt{\Delta}(ACB'^2 - A'C'B^2)$$
$$= (AA' - 3BB' + 2CC')^{2k+1}(ACB'^2 - A'C'B^2),$$

ce qui est une fonction de degré impairement pair, quel que soit l'entier k, par rapport aux coefficients de F, et par suite par rapport à ceux de f, comme on la reconnait avec une legère attention. Mais il nous faut encore approfondir la nature de ces quatre quantités,

$$AA',\ BB',\ CC',\ ACB'^2 - A'C'B^2,$$

qui viennent s'offrir comme éléments simples dans l'expression générale des invariants des formes du cinquième degré. C'est l'objet des considérations qui vont suivre.

Section V.—Des Covariants Similaires.

Revenant au cas général des formes de degré quelconque $f(x, y)$ qui ont un covariant quadratique, soit comme plus haut

$$F = AX^m + mBX^{m-1}Y + \ldots + mB'X^{m-1}Y + A'Y^m,$$

la transformée à laquelle nous avons donné le nom de forme canonique. Par définition même, le covariant quadratique de F sera simplement $XY\sqrt{\Delta}$, cela posé, nous réunirons par la dénomination commune de covariants similaires de f ceux qui jouissent de cette propriété, qu'en y faisant la substitution par laquelle f devient F, leurs coefficients reproduisent toujours à un facteur numérique près, les quantités $A, B, \ldots B', A'$, multipliées par une puissance de $\sqrt{\Delta}$. Cette définition dépend essentiellement du covariant quadratique en x et y, qu'on prend pour base de la réduction à la forme canonique, de sorte qu'on parviendra à un groupe différent de covariants similaires, en employant pour la réduction à la forme canonique, un covariant quadratique en x et y, mais d'un autre degré par rapport aux coefficients. Pour fixer les idées, nous ne considerons que les groupes se rapportant aux covariants quadratiques dont nous avons en commençant établi l'existence par la loi de réciprocité, et nous en donnerons une première série, en nous fondant sur ce théorème :

Soient $\phi(x, y)$ et $\psi(x, y)$ deux covariants quelconques de f, le degré du second étant supposé non-inférieur à celui du premier, en faisant

$$\phi(y, -x) = \alpha x^p + p\beta x^{p-1}y + \ldots + p\beta' xy^{p-1} + \alpha' y^p,$$

le forme

$$\chi = \alpha \frac{d^p\psi}{dx^p} + p\beta \frac{d^p\psi}{dx^{p-1}dy} + \ldots + p\beta' \frac{d^p\psi}{dx\,dy^{p-1}} + \alpha' \frac{d^p\psi}{dy^p},$$

sera encore un covariant de f.*

Pour appliquer ce théorème, nous prendrons $\psi = f$, et nous supposerons ϕ une puissance du covariant quadratique, il viendra alors cette série

$$\sqrt{\Delta}\,\frac{d^2F}{dx\,dy}, \quad \Delta\,\frac{d^4F}{dx^3dy^3}, \quad \Delta\sqrt{\Delta}\,\frac{d^6F}{dx^5dy^5}, \quad \text{etc.},$$

qui aboutit à un invariant, si le degré de F est pair, et à un covariant linéaire si ce degré est impair. Ce covariant linéaire s'évanouit identiquement dans le cas des formes cubiques, car on a alors

$$F = AX^3 + A'Y^3,$$

mais ce cas excepté, il existe bien effectivement. Prenons pour exemple les formes du cinquième degré, le covariant sera alors $\Delta(CX + C'Y)$, et en supposant $C = 0$, $C' = 0$, on ne satisfait plus aux deux relations

$$AB' - 4BC' + 3C^2 = 0, \quad A'B - 4B'C + 3C'^2 = 0,$$

qui seules existent entre les coefficients de F. J'insiste sur ce point en raison de la grande importance des covariants linéaires pour la théorie arithmétique des formes de degrés impairs, dans laquelle si comme j'essayerai de le faire voir, ils jouent un rôle capital. Mais jusqu'à présent nous n'avons obtenu qu'un petit nombre de covariants similaires; par le lemme suivant nous verrons qu'il en existe une infinité. *Nommons comme précédemment $\phi(x, y)$ et $\psi(x, y)$ deux covariants quelconques de la forme proposée f; quelque soient les constantes U et V, la forme suivante*

$$\phi\left(Ux - V\frac{d\psi}{dy}, \quad Uy + V\frac{d\psi}{dx}\right),$$

sera de nouveau un covariant de f.

Nous ferons usage de ce lemme en supposant ϕ l'un quelconque des covariants similaires précédemment obtenus, et prenant pour ψ le covariant quadratique. Désignant alors par $\Phi(X, Y)$ le transformée de ϕ par la substitution

* En supposant $\phi = \psi = f$, χ s'évanouit identiquement si le degré de f est impair, et reproduit si le degré est pair l'invariant quadratique de M. Cayley, que nous trouvons ainsi par une voie nouvelle et très simple.

canonique, celle du covariant quadratique ψ étant dans le même cas $XY\sqrt{\Delta}$, on voit que les coefficients des termes en U et V, dans la forme

$$\Phi\{X(U-V\sqrt{\Delta}),\quad Y(U+V\sqrt{\Delta})\},$$

seront effectivement d'après notre définition des covariants similaires. Maintenant chacun d'eux par l'application repetée du même principe et de celui qui est fondé sur la différentiation, donnera évidemment naissance à une infinité d'autres, tous compris dans la même forme analytique simple, que nous allons indiquer d'une manière plus précise. Soit pour abréger l'écriture, d'après la notation ingénieuse de M. Cayley,

$$F=(A,\ B,\ C,\ \ldots\ C',\ B',\ A')(X,\ Y)^m,$$

de sorte que la première parenthèse renferme dans leur ordre, les coefficients de la forme ; donc les covariants similaires du même degré que F et qui résultent des méthodes précédentes, seront de la forme

$$\Phi=\sqrt{(\Delta^{2k+1})}\,(\alpha A,\beta B,\gamma C,\ldots\ -\gamma C',-\beta B',-\alpha A')(X,Y)^m,$$

ou de la suivante

$$\Phi_1=\Delta^{k_1}(\alpha A,\beta B,\gamma C,\ldots\ \gamma C',\beta B',\alpha A')(X,Y)^m,$$

les quantités α, β, γ, etc., étant des constantes numériques. Les autres de degrés $m-2$, $m-4$, etc.... sont de la forme $\frac{d^{2i}\Phi}{dX^i dY^i}$, $\frac{d^{2i}\Phi_1}{dX^i dY^i}$. Cette remarquable simplicité d'expression que prennent par la substitution canonique une multitude de covariants de la forme f, qu'il eut été impossible d'obtenir jamais en fonction explicite des coefficients de cette forme, justifient ce me semble, l'idée nouvelle des formes canoniques que j'introduis ici.

Section VI.—Recherches ultérieures sur les Invariants des Formes du Cinquième Degré.

Notre point de départ, sera ce théorème auquel conduisent immédiatement les considerations précédentes : soient ϕ et ϕ_1 les covariants de f, qui deviennent respectivement par la substitution canonique les expressions désignées cidessus par Φ et Φ_1, ces covariants étant du même degré, on obtiendra un invariant en mettant dans $\phi(y_1,-x)$, $\frac{d^m\phi_1}{dx^i dy^{m-i}}$, au lieu de $x^i y^{m-i}$, d'après un théorème enoncé plus haut.

Cela étant, cet invariant exprimé par les coefficients $A, B, C, \ldots$ de la forme canonique, sera

$$I = \Delta^{k+\frac{1}{2}}\sqrt{\Delta}\left(\alpha^2 AA' - m\beta^2 BB' + \frac{m.m-1}{2}\gamma^2 CC' - \text{etc.}\ldots\right).$$

De là se tire une méthode très simple, dont nous allons faire l'application aux formes du cinquième degré pour exprimer les quantités AA', BB', CC', etc., au moyen des coefficients de la forme proposée. Posons

$$f = (a, b, c, c', b', a')\,(x, y)^5,$$

le covariant quadratique, dans le même système de notation, sera

$$\phi = (ab' - 4bc' + 3c^2,\ aa' - 3bb' + 2cc',\ a'b - 4b'c + 3c'^2)\,(x^2, xy, y^2),$$

et en faisant

$$f\left(Ux - V\frac{d\phi}{dy},\ Uy + V\frac{d\phi}{dx}\right) = (f, f_1, f_2, f_3, f_4, f_5)\,(U, V)^5,$$

les diverses formes f, f_1, f_2, etc. seront un groupe de covariants similaires, que nous allons employer à la composition de trois invariants I_0, I_1, I_2, à savoir

I_0 en employant f avec f_1,
I_1 ..... id. ..... f_3,
I_2 ..... id. ..... f_5.

Ces invariants seront respectivement des degrés 4, 8, 12, car il est aisé de voir que les covariants f_1, f_3, f_5, sont des degrés 3, 5 et 7. Cela posé, nommons F_1, F_3, F_5 leurs transformées respectives par la substitution canonique, on aura très facilement ces expressions,

$$F = (A, B, C, C', B', A')\,(X, Y)^5,$$
$$F_1 = (A, \tfrac{3}{5}B, \tfrac{1}{5}C, -\tfrac{1}{5}C', -\tfrac{3}{5}B', -A')\,(X, Y)^5\sqrt{\Delta},$$
$$F_3 = (A, -\tfrac{1}{5}B, -\tfrac{1}{5}C, \tfrac{1}{5}C', \tfrac{1}{5}B, -A')\,(X, Y)^5\sqrt{\Delta^3},$$
$$F_5 = (A, -B, C, -C', B, -A')\,(X, Y)^5\sqrt{\Delta^5},$$

d'où l'on déduira

$$(I)\quad \begin{cases} I_0 = 2\sqrt{\Delta}\,(AA' - 3BB' + 2CC') = 2\Delta, \\ I_1 = 2\sqrt{\Delta^3}\,(AA' + BB' - 2CC'), \\ I_2 = 2\sqrt{\Delta^5}\,(AA' + 5BB' + 10CC'). \end{cases}$$

Voici donc un moyen d'obtenir explicitement par les coefficients de f, les trois quantités AA', BB', CC', car le

déterminant relatif aux équations précédentes, est différent de zéro, et égal à 2^5. Mais les expressions générales données § 4e contiennent en outre le quantité $ACB'^2 - A'C'B^2$, que nous obtiendrons de la manière suivante. Considerons les covariants similaires ayant pour transformées canoniques

$$\sqrt{\Delta}\,\frac{d^2F}{dXdY} \text{ et } \Delta\,\frac{d^4F}{dX^2dY^2}$$

en formant le cube du dernier, on parviendra aux deux formes

$$\sqrt{\Delta}(B,\ C,\ C',\ B')\,(X,Y)^3,$$

et

$$\Delta^3(C^3,\ C^2C',\ CC'^2,\ C'^3)\,(X,Y)^3,$$

d'où l'on tire toujours par le même principe, l'invariant du 18e degré que nous nommerons I_3,

$$I_3 = \Delta^3\sqrt{\Delta}(BC'^3 - B'C^3).$$

Mais par les relations fundamentales

$$AB' - 4BC' + 3C^2 = 0, \quad A'B - 4B'C + 3C'^2 = 0,$$

on obtient facilement

$$3(BC'^3 - B'C^3) = ACB'^2 - A'C'B^2,$$

d'où enfin

$$ACB'^2 - A'C'B^2 = 3\,\frac{I_3}{\Delta^3\sqrt{\Delta}}.$$

Voici donc, d'après les formules du § 4e, la conclusion de notre théorie pour les formes du cinquième degré.

1°. *L'expression la plus générale des Invariants de ces formes dont le degré* $\mu \equiv 0 \pmod 4$ *est*

$$I = F\left(\sqrt{\Delta},\ \frac{I_1}{\Delta\sqrt{\Delta}},\ \frac{I_2}{\Delta^2\sqrt{\Delta}}\right),$$

F étant une fonction homogène du degré $\frac{1}{2}\mu$.

2°. *L'expression la plus générale des Invariants dont le degré* $\mu \equiv 2 \pmod 4$ *est*

$$I' = \frac{I_3}{\Delta^3\sqrt{\Delta}}\,F_1\left(\sqrt{\Delta},\ \frac{I_1}{\Delta\sqrt{\Delta}},\ \frac{I_2}{\Delta^2\sqrt{\Delta}}\right),$$

F_1 *étant une fonction homogène de degré* $\frac{1}{2}\mu - 2$.

Ainsi un invariant quelconque, ou au moins son produit par une puissance de Δ, est une fonction rationnelle et entière des invariants fondamentaux Δ, I_1, I_2, I_3, des degrés 4, 8, 12,

et 18. Car les fonctions F et F_1 étant homogènes, on peut écrire

$$I = \frac{1}{\Delta^{\frac{1}{4}\mu}} F(\Delta^3, \Delta I_1, I_2) \text{ et } I' = \frac{I_2}{\Delta^{\frac{1}{4}(6\mu-6)}} F_1(\Delta^3, \Delta I_1, I_2).$$

Nous voyons par la se révéler un caractére essentiel des formes de degré superieur au quatrième, et qui consiste en ce que les invariants ne peuvent en général s'exprimer en fonction rationnelle d'un certain nombre d'entre eux supposés algébriquement indépendants. M. Cayley, M. Sylvester, et moi avions long-temps pensé qu'en général les invariants des formes de m^e degré devaient s'exprimer par des fonctions entières de $m-2$ d'entre eux, et c'est même ce qui à empêché M. Sylvester de chercher à démontrer la loi de réciprocité dont il avait aussi présumé l'existence, une contradiction necessaire s'étant manifestée entre cette loi et celle du nombre des invariants fondamentaux. Peut-être cependant, s'il m'est permis d'emettre une conjecture sur un sujet si profond et si difficile, doit-on penser qu'il sera possible d'obtenir pour les formes d'un degré donné, un petit nombre de groupes d'invariants fondamentaux, types d'autant de séries générales dont l'ensemble comprendrait tous les invariants possibles. C'est ainsi par exemple que l'invariant du 18e degré que nous venons d'obtenir pour les formes du cinquième degré s'offre comme le type de tous les invariants de degré impairement pair de ces formes. Sur ce sujet nous allons encore présenter quelques observations.

Section VII.—Recherche particulière sur l'Invariant I_3 du 18e Degré.

Je me propose de faire voir que le carré de I_3 est non seulement une fonction rationnelle, mais même une *fonction entière* des trois invariants nommés Δ, I_1 et I_2. Soit à cet effet $I_2 - 2I_1\Delta + \Delta^3 = 24J_2$ et $I_1 - \Delta^2 = 8J_1$, j'adopterai pour invariants fondamentaux J_1 et J_2 au lieu de I_1 et I_2, pour la commodité des calculs, et les équations (I) du § 6e donneront ces expressions très simples

$$CC' = \frac{J_2}{\sqrt{\Delta^5}}, \quad BB' = \frac{J_2 + J_1\Delta}{\sqrt{\Delta^5}}, \quad AA' = \frac{J_2 + 3J_1\Delta + 2\Delta^3}{\sqrt{\Delta^5}}.$$

Cela posé, nous partirons de la rélation suivante,

$$16(ACB'^2 - A'C'B^2)^2 = (AA'BB' - 16BB'CC - 9C^2C'^2)^3 - 24^2BB'C^3C'^3,$$

qu'on trouvera identique, en vertu des équations fondamentales qu'on a entre les coefficients de la forme canonique savoir

$$AB' - 4BC' + 3C^2 = 0, \quad A'B - 4B'C + 3C'^2 = 0.$$

On peut effectivement d'abord l'écrire ainsi,

$$24^2 BB'C^3C'^3 = (AA'BB' - 16BB'CC' - 9C^2C'^2)^2 - 16(ACB'^2 - A'C'B^2)^2,$$

ou en de composant en produit la différence des carrés,

$$24^2 BB'C^3C'^3 = \{AA'BB' - 16BB'CC' - 9C^2C'^2 + 4(ACB'^2 - A'C'B^2)\} \{AA'BB' - 16BB'CC' - 9C^2C'^2 - 4(ACB'^2 - A'C'B^2)\}.$$

Maintenant les équations

$$3C^2 = 4BC' - AB', \quad 3C'^2 = 4B'C - A'B,$$

donneront si on les multiplie membre à membre,

$$9C^2C'^2 = 16BB'CC' + AA'BB' - 4(ACB'^2 + A'C'B^2),$$

et en substituant cette valeur de $C^2C'^2$ dans chacun des facteurs, on verra le premier devenir

$$8ACB'^2 - 32BB'CC' = 8B'C(AB' - 4BC') = -24B'C^3,$$

et le second se réduire d'une manière semblable à

$$8A'C'B^2 - 32BB'CC' = 8BC'(A'B - 4BC') = -24BC'^3,$$

d'on suit l'identité annoncée. L'expression du carré de $ACB'^2 - A'C'B^2$, étant alors ramenée à ne plus dépendre que des quantités AA', BB', CC', on trouvera par la substitution des valeurs de ces quantités, un fonction des invariants Δ, J_1, J_2, et en chassant le dénominateur

$$16\Delta^{10}(ACB'^2 - A'C'B^2)^2 = (-24J_2^2 - 12J_1J_2\Delta + 3J_1^2\Delta^2 + 2J_2\Delta^3 + 2J_1\Delta^4)^2 - 24^2J_2^3(J_2 + J_1\Delta).$$

Or il arrive que le second membre contient en facteur Δ^3, de sorte qu'en supprimant ce facteur il viendra

$$16\Delta^7(ACB'^2 - A'C'B^2)^2 = 16I_3^2 = -24J_2(2J_2^2 + 3J_1^3) + 3\Delta J_1(3J_1^3 - 32J_2^2) - 12\Delta^2J_1^2J_2 + 4\Delta^3(J_2^2 + 3J_1^3) + 4\Delta^4J_1J_2 + 4J_2^2\Delta^5,$$

ce qui est une fonction entière des trois invariants fondamentaux, Δ, J_1, et J_2.

SECONDE PARTIE.

Depuis que la première partie de ces recherches à été terminée, encouragé par la manière si bienveillante dont elles ont été accueillies par mon ami M. Sylvester, j'ai repris avec une nouvelle ardeur l'étude algébrique des formes du cinquième degré, et je vais y consacrer cette seconde partie de mon travail, en réservant en dernier lieu, les considérations arithmétiques que j'ai annoncées dans l'introduction. C'est sur une notion analytique nouvelle, celle des formes-types, qui sera tout-à-l'heure exposée en détail, que se fondent les résultats nouveaux que j'ai obtenus. Cette notion est essentiellement propre aux formes de degrés impairs, avec la seule exception des formes cubiques qui y échappent comme un cas singulier. Pour les formes de degrés pairs il existe quelque chose d'analogue, mais qui jusqu'à présent ne s'est présenté à moi, que d'une manière plus compliquée. Aussi en parlerai-je seulement pour remarquer que les formes biquadratiques font alors exception, de sorte que les formes des premiers quatre degrés, pour des raisons diverses, doivent être considerées comme présentant des cas singuliers dans les théories générales qui ont pour objet les fonctions homogènes à deux indéterminées. C'est donc au seul point de vue algébrique, un champ plus vaste et plus fécund de recherches, qui s'ouvre à partir des formes de cinquième degré, où l'on voit apparâitre le rôle curieux d'éléments analytiques, qui n'existent pas pour les formes de degrés inférieurs. D'ailleurs c'est dans les méthodes simples et faciles qui se présentent dans cette étude, où est l'avenir de la science algébrique, car elle seule peut donner les éléments qui distinguent et caracterisent les divers modes d'existence des racines des équations générales de tous les degrés. J'espére que cette dernière considération recevera sa sanction de ce que nous allons développer en particulier sur les formes du cinquième degré.

Section I.—Des Formes-types.

La notion des formes-types repose sur l'existence des covariants linéaires, dont il à été déja fait mention précédemment, et qu'on obtient de la manière suivante. Soit en employant la notation de Mr. Cayley,

$$f = (a, b, c, \ldots c', b', a')\,(x, y)^m,$$

une forme de degré impair,

$$\theta = \{ab' - (m-1)\,bc' + \ldots,\quad aa' - (m-2)\,bb' + \ldots,$$
$$ba' - (m-1)\,b'c + \ldots\}\,(x^2, xy, y^2),$$

le covariant quadratique de f, et Δ l'invariant de θ. Nommons S le substitution au déterminant *un* et aux variables X, Y, qui transforme θ en $\sqrt{\Delta}.XY$, cette même substitution faite dans la proposée f, donnera ce que nous avons nommé la transformée canonique

$$F = (A, B, C, \ldots C', B', A')(X, Y)^m.$$

Ainsi le caractére essentiel de la forme canonique F, est que le covariant Θ analogue à θ, se réduise à $\sqrt{\Delta}\,XY$; les coefficients A, B, etc. ... sont donc liés par les relations

$$AB' - (m-1)\,BC' + \ldots = 0, \quad A'B - (m-1)\,B'C + \ldots = 0.$$

Ceci rappelé, voici comment s'obtient un covariant linéaire λ de la forme f. Elevons θ à la puissance $\frac{1}{2}(m-1)$, ce qui donnera un covariant du degré $m-1$ en x et y, puis mettons y et $-x$ au lieu de x et y; cela fait, en remplaçant un terme quelconque $x^\alpha y^\beta$, par $\dfrac{d^{m-1}f}{dx^\alpha dy^\beta}$, on obtiendra, comme on sait, encore un covariant de f, et ce covariant sera bien du premier degré. Mais il est essentiel d'établir qu'il ne s'évanouit pas identiquement. Soit à cet effet Λ, la transformée de λ, par la substitution S, on aura

$$\Lambda = \sqrt{(\Delta^{\frac{1}{2}(m-1)})}\,\frac{d^{m-1}F}{dx^{\frac{1}{2}(m-1)}dy^{\frac{1}{2}(m-1)}};$$

supposant donc

$$\frac{d^{m-1}F}{dx^{\frac{1}{2}(m-1)}dy^{\frac{1}{2}(m-1)}} = GX + G'Y,$$

Λ ne pourra s'évanouir identiquement qu'autant qu'on aura $G = 0$, $G' = 0$, mais ces relations ne vérifient pas les équations (1), sauf le cas des formes cubiques. Dans ce cas en effet, ayant $F=(A,B,B',A')(X,Y)^3$, Λ sera $3\sqrt{\Delta}(BX+B'Y)$, mais les relations (1)

$$AB' - B^2 = 0, \quad A'B - B'^2 = 0,$$

exigeront que $B = 0$, $B' = 0$. On en déduit effectivement

$$AA'BB' = B^2B'^2,$$

d'ou
$$BB'(AA' - BB') = 0.$$

Si donc le produit BB' n'est pas supposé nul, il faut qu'on

ait $AA' - BB' = 0$, ce qui conduit à la conséquence absurde que l'invariant

$$(AA' - BB')^3 - 4(AB' - B^2)(A'B - B'^2),$$

de la forme cubique est égal à zero. Les covariants linéaires n'ont donc d'existence effective qu' à partir des formes du cinquième degré

$$f = (a, b, c, c', b', a')(x, y)^5,$$

mais pour ces formes il y en aura un nombre infini, dont les degrés par rapport aux coefficients a, b, c, etc. seront la série des nombres impairs 5, 7, 9, etc.

Le covariant du 5^e ordre sera celui que nous venons d'obtenir et dont la transformée par la substitution S est $10\Delta(CX + C'Y)$, le covariant du 7^e ordre résultera du précédent, en y remplaçant x et y, par $\frac{d\theta}{dy}$ et $-\frac{d\theta}{dx}$, d'autres pourront s'obtenir en multipliant les précédents par des invariants de f. En se bornant à prendre pour multiplicateur une puissance de Δ, on obtiendra ainsi des covariants linéaires dont les degrés par rapport aux coefficients de f, seront les nombres $4n + 5$ et $4n + 7$, c. à. d. la série des entiers impairs à commencer par cinq. Nous en concluerons par la loi de réciprocité que toutes les formes dont les degrés sont des nombres impairs à partir du cinq, possédent un covariant linéaire du cinquième degré par rapport à leurs coefficients, et il est très facile d'établir qu'elles n'en possédent pas dont les degrés soient au dessous de cette limite. Mais pour abréger j'omettrai ce détail, et j'arrive immediatement à la définition des formes-types. Soient à cet effet λ et λ_1 deux covariants linéaires distincts pour une même forme f; designons par Σ la substitution

$$\lambda = \xi, \quad \lambda_1 = \eta,$$

et par Φ la transformée de f en ξ et η. Je dis que les coefficients de cette forme Φ, seront tous des invariants de f.

Pour le démontrer voyons ce deviennent les opérations précédentes en prenant pour point de départ une forme f', transformée de f par une substitution quelconque S. Soit δ le déterminant relatif à cette substitution S, λ' et λ_1' les covariants analogues à λ et λ_1. En multipliant par ordres puissances convenables de δ, p. ex. δ^α et δ^β, chacune des équations

$$\lambda' = \xi,$$
$$\lambda_1' = \eta,$$

il résulte de la nature même des covariants, que les premiers membres pourront alors être censés provenir du résultat de la substitution S dans λ et λ'. Ainsi par rapport aux quantités $\delta^{\alpha}\xi$, $\delta^{\beta}\eta$, la substitution Σ' analogue à Σ, sera $\Sigma' = \Sigma S$, et son inverse qu'il faudra effectuer dans f, sera $\Sigma'^{-1} = S^{-1}\Sigma$. Or on voit qu'en effectuant en premier la substitution S^{-1}, f' redevient f, et qu'en faisant ensuite la substitution Σ on est ramené, précisément à la forme ϕ, par rapport aux indéterminées $\delta^{\alpha}\xi$, $\delta^{\beta}\eta$. De la résulte que les coefficients des formes ϕ, relatives à f, et à une transformée de f, ne différent que par des facteurs qui seront des puissances du déterminant de la substitution, ces coefficients seront donc des invariants de f; et c'est pour cette raison que nous donnons à ϕ le dénomination de forme-type.

Section II.—Calcul de la Forme-type du Cinquième Degré.

La définition que nous venons de donner, ne spécifie pas les covariants linéaires qu'il faut employer dans la substitution qui conduit aux formes-types, il suffit que ces covariants soient bien distincts c. à. d. que le déterminant relatif à la substitution effectuée soit différent de zéro. Mais dans le cas des formes du cinquième degré, que nous allons étudier nous ferons choix des deux covariants linéaires les plus simples, qui sont respectivement du cinquième et du septième degré par rapport aux coefficients de la forme proposée. En effectuant dans ces covariants la substitution S qui transforme f dans la forme canonique F, ils deviendront

$$\lambda = 10\Delta(CX + C'Y), \quad \lambda_1 = 10\Delta\sqrt{\Delta}(CX - C'Y),$$

expressions très simples, qui nous conduisent à faire le calcul de la forme-type, en opérant sur la transformée canonique F, ce qui est permis, puisqu'on parviendra identiquement au même résultat, en prenant pour point de depart toute transformée de f, par une substitution au déterminant *un*. Cela posé, ayant,

$$F = (A, B, C, C', B', A')(X, Y)^5,$$

nous ferons en supprimant un facteur numérique,

$$\Delta(CX + C'Y) = \xi, \quad \Delta\sqrt{\Delta}(CX - C'Y) = \eta,$$

et si nous représentons la transformée en ξ et η par,

$$\phi = (A, B, C, C', B', A')(\xi, \eta)^5,$$

il viendra ces expressions,

$$A = \frac{1}{(2CC'\Delta)^5}\{AC'^5 + A'C^5 + 5BCC'^4 + 5B'C'C^4 + 20C^3C'^3\},$$

$$B = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta}}\{AC'^5 - A'C^5 + 3BCC'^4 - 3B'C'C^4\},$$

$$C = \frac{1}{(2CC'\Delta)^5\Delta}\{AC'^5 + A'C^5 + BCC'^4 + B'C'C^4 - 4C^3C'^3\},$$

$$C' = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta^3}}\{AC'^5 - A'C^5 - BCC'^4 + B'C'C^4\},$$

$$B' = \frac{1}{(2CC'\Delta)^5\Delta^2}\{AC'^5 + A'C^5 - 3BCC'^4 - 3B'C'C^4 + 4C^3C'^3\},$$

$$A' = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta^5}}\{AC'^5 - A'C^5 - 5BCC'^4 + 5B'C'C^4\}.$$

D'après les théorèmes donnés au commencement de ces recherches, on reconnait tout de suite qu'elles sont bien comme nous l'avons annoncé, des invariants de la forme proposée f, et qu'elles s'exprimeront rationnellement par les fonctions que nous avons nommées Δ_1, J_2, J_3, et par l'invariant du 18e ordre Γ. Mais ici se présente cette circonstance importante, qu'elles contiendront en dénominateur le seul invariant J_3, sans qu'on y voie figurer Δ, comme on pouvait s'y attendre d'après la théorie générale. Pour le faire voir, rappelons d'abord ces rélations qui existent entre les invariants et les coefficients de la forme canonique savoir :

$$AA' = \frac{1}{\sqrt{\Delta^5}}(J_3 + 3\Delta J_2 + \Delta^3),$$

$$BB' = \frac{1}{\sqrt{\Delta^5}}(J_3 + \Delta J_2),$$

$$CC' = \frac{1}{\sqrt{\Delta^5}}J_3,$$

$$ACB'^2 - A'C'B^2 = \frac{1}{\sqrt{\Delta^7}}I.$$

Nous en déduirons les valeurs des quantités $AC'^5 \pm A'C^5$ et $BC'^3 \pm B'C^3$, qui figurent dans les coefficients A, B, etc., par les équations suivantes :

$$36(AC'^5 + A'C^5)$$
$$= (AA' + 16CC')(AA'BB' + 16BB'CC' - 9C^3C'^2) - 64AA'BB'CC',$$

$$12(BC'^3 + B'C^3) = -AA'BB' + 16BB'CC' + 9C^3C'^2,$$

$$9(AC'^5 - A'C^5) = (16CC' - AA')(ACB'^2 - A'C'B^2),$$

$$3(BC'^3 - B'C^3) = ACB'^2 - A'C'B^2.$$

Ces équations deviennent effectivement identiques, en vertu des rélations fondamentales qui lient les coefficients de la forme canonique savoir,

$$AB' - 4BC' + 3C^2 = 0, \quad A'B - 4B'C + 3C'^2 = 0.$$

Quant à la méthode très facile par laquelle on les obtient, je pense pouvoir la supprimer pour abréger, car elle se présentera d'elle-même au lecteur qui se sera bien penetré des principes de ces recherches. On en déduit

$$9A' = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta^5}}(CC' - AA')(ACB'^2 - A'C'B^2)$$
$$= -I\frac{\Delta^2 + 3J_2}{2^5J_3^5},$$

$$9C' = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta^3}}(13CC' - AA')(ACB'^2 - A'C'B^2)$$
$$= -I\frac{\Delta^3 + 3\Delta J_2 - 12J_3}{2^5J_3^5},$$

$$9B = \frac{1}{(2CC'\Delta)^5\sqrt{\Delta}}(25CC' - AA')(ACB'^2 - A'C'B^2)$$
$$= -I\frac{\Delta^4 + 3\Delta^2J_2 - 24\Delta J_3}{2^5J_3^5},$$

Le calcul des trois autres coefficients est un peu plus difficile et donne pour resultats,

$$36B' = \frac{1}{(2CC'\sqrt{\Delta})^5\Delta^2}(-81C^3C'^3 + 112BB'C^2C'^2 - 9AA'C^2C'^2$$
$$- 23AA'BB'CC' + A^2A'^2BB')$$
$$= \frac{1}{2^5J_3^5}(\Delta^5J_2 + \Delta^4J_3 + 6\Delta^3J_2^2 - 15\Delta^2J_2J_3 - 3\Delta(10J_3^2 - 3J_2^3) - 54J_3J_2^2),$$

$$36C = \frac{1}{(2CC'\sqrt{\Delta})^5\Delta}(-261C^3C'^3 + 304BB'C^2C'^2 - 9AA'C^2C'^2$$
$$- 35AA'BB'CC' + A^2A'^2BB')$$
$$= \frac{1}{2^5J_3^5}(\Delta^6J_2 + \Delta^5J_3 + 6\Delta^4J_2^2 - 27\Delta^3J_2J_3 - 3\Delta^2(14J_3^2 - 3J_2^3)$$
$$- 90\Delta J_3J_2^2 + 144J_2J_3^2),$$

$$36A = \frac{1}{(2CC'\Delta)^5}(711C^3C'^3 + 496BB'C^2C'^2 - 9AA'C^2C'^2$$
$$- 47AA'BB'CC' + A^2A'^2BB')$$
$$= \frac{1}{2^5J_3^5}(\Delta^7J_2 + \Delta^6J_3 + 6\Delta^5J_2^2 - 39\Delta^4J_2J_3 - 9\Delta^3(6J_3^2 - J_2^3)$$
$$- 126\Delta^2J_3J_2^2 + 288\Delta J_2J_3^2 + 1152J_3^3).$$

Ainsi il est démontré par le calcul, que les coefficients de

la forme-type, deviennent des fonctions entières des quatre invariants fondamentaux lorsqu'on les multiplie par J_3^5, et c'est la une remarque qui nous conduira plus loin à des conséquences importantes.

Section III.—L'équation générale du Cinquième Degré est ramenée à ne dependre que de deux Paramètres.

Ce résultat suit immédiatement de l'expression de la forme-type que nous venons d'obtenir. Qu'on fasse en effet $K = \frac{J_2}{\Delta^2}$, $K' = \frac{J_3}{\Delta^3}$, et on pourra écrire

$$\phi = \frac{\Delta^6\sqrt{\Delta}}{(2J_3)^5}(A_0\sqrt{\Delta^5}, B_0\sqrt{\Delta^4}, C_0\sqrt{\Delta^3}, C_0'\sqrt{\Delta^2}, B_0'\sqrt{\Delta}, A_0')(\xi, \eta)^5,$$

A_0, B_0, etc., désignant respectivement ce que deviennent les numérateurs dans A, B, etc., quand y remplace Δ, J_2, J_3, par 1, K, K'. Nommons de même I_0 ce que devient alors l'invariant I, il est clair qu'il suffit de mettre $\eta I_0\sqrt{\Delta}$, au lieu de η, pour ramener l'équation $\phi = 0$, à contenir seulement les deux paramètres K et K'. Et s'il arrive que Δ soit une quantité négative, la réduction sera aussi bien obtenue, en remplaçant η par $\eta I_0\sqrt{(-\Delta)}$, c'est la seule quantité irrationnelle qui figure dans la substitution, et il est aisé de voir que l'irrationnelle $I_0\sqrt{\Delta}$ disparaitra dans l'équation transformée, de sorte que K et K', entreront rationnellement dans le résultat. L'équation a laquelle nous méne ainsi la notion des formes-types, n'a pas la simplicité apparente de la réduite de l'équation du cinquième degré qu'a obtenue Jerrard, mais elle met in évidence les fonctions des coefficients dont dépend essentiellement la nature des racines, et tandis que l'ingénieuse découverte du géomètre Anglais, est restée jusqu'ici stérile, nous allons pouvoir immédiatement tirer d'importantes conséquences de notre transformée.

Section IV.— Les Invariants de tous les Degrés des Formes du Cinquième Degré, sont des fonctions entières des quatre invariants fondamentaux Δ, J_2, J_3, *et* I.

Dans la première partie de ces récherches, j'ai obtenue pour les invariants dont le degré est $\equiv 0$ ou $\equiv 2$ (mod. 4) les expressions générales

$$\frac{\Theta_0(J_3, \Delta J_2, \Delta^3)}{\Delta^\mu} \text{ et } \frac{I\Theta_2(J_3, \Delta J_2, \Delta^3)}{\Delta^\mu},$$

où entre en dénominateur une puissance de Δ, mais on peut aller plus loin et parvenir à des expressions entières, par la considération de la forme-type. En effet, ϕ étant une transformée de f, par une substitution linéaire au déterminant $\frac{\Delta}{2J_3}$, comme il est aisé de le voir, tout invariant de f, s'exprime au moyen d'une fonction semblable des coefficients de ϕ, multiplié par une certaine puissance de J_3. Mais ces coefficients de la forme-type, sont comme nous l'avons établi, des fonctions entières des invariants fondamentaux divisés par une puissance de J_3, donc déjà, tout invariant de la forme proposée est une fonction entière des invariants fondamentaux, au moins une pareille fonction divisée par une puissance de J_3. Distinguant maintenant les deux cas ou le degré des invariants est $\equiv 0$ ou $\equiv 2 \pmod 4$, nous reconnaitrons bien aisément que l'expression générale

$$\frac{F(\Delta, J_2, J_3, I)}{J_3^{\nu}},$$

ou F est une fonction entière, se réduit dans le premier, à la forme $\frac{H_0(\Delta, J_2, J_3)}{J_3^{\nu}}$, et dans le second à la forme $\frac{IH_1(\Delta, J_2, J_3)}{J_3^{\nu}}$, H_0 et H_1 étant pareillement des fonctions entières. Cela suit en effet de ce que le carré et les puissance paires de l'invariant I du 18^e degré, s'expriment en fonction entière de Δ, J_2, et J_3. Voici donc, par exemple, pour les invariants dont le degré est multiple de 4, deux expressions differentes, qui doivent être égales

$$\frac{\Theta_0(J_3, \Delta J_2, \Delta^3)}{\Delta^{\mu}} \text{ et } \frac{H_0(\Delta, J_2, J_3)}{J_3^{\nu}};$$

or les trois quantités Δ, J_2, J_3, qui y figurent, n'ont entre elles aucune rélation, et doivent être considerés comme absolument indépendantes, l'égalité

$$\frac{H_0(\Delta, J_2, J_3)}{J_3^{\nu}} = \frac{\Theta_0(J_3, \Delta J_2, \Delta^3)}{\Delta^{\mu}},$$

entraîne donc que H_0 est divisible par J_3^{ν} et Θ par Δ^{μ}, *c. à. d.*, que les invariants en question, s'expriment en fonction entière de Δ, J_2, et J_3. Quand au second cas ou le degré est $\equiv 2 \pmod 4$, il se traite tout-à-fait de même, cas il conduit à l'égalité

$$\frac{I\Theta_1(J_3, \Delta J_2, \Delta^3)}{\Delta^{\mu}} = \frac{IH_1(\Delta, J_2, J_3)}{J_3^{\nu}},$$

qui après la suppression du facteur I, coincide avec celle qu'on vient d'obtenir, ainsi donc en général, tout invariant d'une forme du cinquième degré, dont le degré par rapport aux coefficients est $\equiv 0 \pmod 4$, est une fonction entière de Δ, J_2, J_3, et tout invariant dont le degré est $\equiv 2$, est la produit d'une pareille fonction multipliée par l'invariant I du 18e degré. Les expressions suivantes

$$\Sigma\alpha.\Delta^{i}J_2^{i'}J_3^{i''}, \quad I\Sigma\alpha.\Delta^{i}J_2^{i''}J_3^{i''},$$

ou les quantités α sont numériques, représentent donc tous les invariants des formes du cinquième degré, d'où l'on voit qu'il existe autant d'invariants linéairement indépendants, d'un degré donné m, qu'il y a de solutions entières et positives de l'une or l'autre de ces équations

$$4i + 8i' + 12i'' = m,$$

$$18 + 4i + 8i' + 12i'' = m.$$

On en conclut par la loi de réciprocité, que les formes d'un degré quelconque m, ont autant d'invariants du cinquième degré par rapport à leurs coefficients, qu'il y a de solutions entières et positives des mêmes équations. Ainsi parmi les formes dont le degré est impairement pair, il faut aller jusqu'au 18e degré pour rencontrer un invariant du cinquième ordre.

Section V.—Recherche particulière sur le Discriminant des Formes du Cinquième Degré.

MM. Cayley et Sylvester nomment, comme on sait, discriminant d'une forme f, le résultat de l'élimination de $\frac{x}{y}$, entre les deux équations homogènes $\frac{df}{dx} = 0$, $\frac{df}{dy} = 0$. On obtient ainsi pour une forme de degré m un invariant de degré $2(m-1)$, qui égalé à zero exprime que f a un facteur linéaire elevé au carré. Dans le cas des formes du cinquième degré, le discriminant est donc un invariant du 8e ordre, et qui d'après la théorie précédente doit être de cette forme $\alpha J_2 + \alpha'\Delta^2$, α et α' étant numériques. Mais nous allons en former l'expression par une méthode particulière et sans supposer les résultats généraux établis dans le précédent §, dont nous voulons offrir ainsi une confirmation dans un cas spécial très important en lui-même. A cet effet, nous nous proposerons généralement d'obtenir

les valeurs des invariants fondamentaux, lorsqu'il existe un facteur linéaire élevé au carré dans la forme proposée, *c. à. d.*, lorsqu'on peut lui donner cette expression

$$f = (0, 0, a, b, c, d)(x, y)^5.$$

En observant qu'on peut mettre $x + ky$, au lieu de x sans que les deux premiers coefficients cessent d'être nuls, disposons de cette quantité k, de manière a faire évanouir le coefficient de xy, dans le covariant quadratique θ. Nous aurons ainsi une transformée

$$f_1 = (0, 0, a_1, b_1, c_1, d_1)(x, y)^5,$$

et il faudra que les nouveaux coefficients vérifient la condition $a_1 b_1 = 0$. Comme nous ne voulons point admettre de facteurs linéaires à la troisième puissance, il faudra faire $b_1 = 0$, et si l'on écrit ainsi f_1 sous la forme

$$f_1 = \left(0, 0, \frac{4}{a^2}, 0, \frac{3}{b^2}, e\right)(x, y)^5,$$

le covariant θ sera

$$\theta = 12\left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right),$$

ce qui nous conduit à remplacer encore $\frac{x}{a}$ et $\frac{y}{b}$ par X et Y. Nous trouverons de le sorte cette transformée des formes à facteur lineaire double

$$pY^5 + q(3XY^4 + 8X^3Y^2),$$

ou p et q sont des constantes quelconques, et qui a pour covariant quadratique

$$\frac{48q^2}{25}(X^2 - Y^2).$$

Cela étant, il suffira de mettre $X + Y$ et $X - Y$, au lieu de X et Y, pour obtenir la transformée canonique, qui sera

$$\begin{aligned} F &= p(X-Y)^5 + q(X-Y)^3(X+Y)\{3(X-Y)^2 + 8(X+Y)^2\}, \\ &= p(X-Y)^5 + q(X-Y)^3(X+Y)(11X^2 + 10XY + 11Y^2), \\ &= (p + 11q, -p - \tfrac{1}{5}q, p - q, -p - q, p - \tfrac{1}{5}q, -p + 11q)(X, Y)^5. \end{aligned}$$

Voici donc en fonction des deux indéterminées p et q, les valeurs suivantes, propres au cas d'un facteur linéaire élevé au carré dans la forme proposée, savoir,

$$AA' = -p^2 + 121q^2, \quad BB' = -p^2 + \tfrac{1}{25}q^2, \quad CC' = -p^2 + q^2,$$
$$ACB'^2 - A'C'B^2 = \tfrac{48}{5}pq(2p^2 + q^2).$$

On en tire

$$\sqrt{\Delta} = \frac{3.2^{10}}{25} q^3, \quad \frac{J_2}{\sqrt{\Delta^3}} = -\frac{3.2^3}{25} q^3, \quad \frac{J_3}{\sqrt{\Delta^5}} = q^2 - p^2,$$

d'où cette conclusion importante,

$$\Delta^2 + 2^7 J_2 = 0.$$

Le discriminant des formes du cinquième degré est donc obtenu, puisque nous avons un invariant du 8e ordre $\Delta^2 + 2^7 J_2$ qui s'évanouit lorsqu'on suppose deux racines égales dans ces formes, et il se présente bien sous la forme valide d'après notre théorie générale. Exprimé par les coefficients de la forme canonique, il a cette valeur,

$$\text{discriminant} = \Delta^2 + 2^7 J_2 = \sqrt{\Delta^3}(AA' + 125BB' - 126CC'),$$

de sorte qu'on a un procédé arithmétique facile, pour calculer dans un cas donné cette fonction si importante. Remarquons encore avant d'aller plus loin, la quantité

$$25\Delta^3 - 2^{11} J_3,$$

qui s'évanouit si la forme proposée contient deux facteurs linéaires différents élevés chacune au carré. Si l'on cherche en effet, la discriminant de la forme cubique

$$\frac{F}{(X-Y)^2} = p(X-Y)^3 + q(X+Y)(11X^2 + 10XY + 11Y^2),$$

on la trouvera abstraction faite d'un facteur numérique égal à $q^2(2p^2+q^2)$ et d'après les relations précédentes, cette valeur s'exprime ainsi

$$25\,\frac{25\Delta^3 - 2^{11} J_3}{3^2.2^{20}\Delta^2}.$$

Dans un instant nous allons reconnaître le rôle important que joue cette quantité.*

Section VI.

Expression par les Invariants Fondamentaux, du nombre des racines réelles et imaginaires de toute équation du Cinquième Degré.

La possibilité d'un pareil résultat est une conséquence immédiate de ces deux propriétés de la forme-type, d'être une transformée par une substitution réelle de la forme

* Dans le cas d'une forme contenant au cube un facteur linéaire, tous les invariants s'évanouissent, comme cela résulte d'un théorème générale donné par mon ami M. Cayley dans le Journal de M. Crelle.

proposée, et d'avoir pour coefficients des invariants. Mais on sent combien il y a loin d'une telle possibilité à un résultat effectif, aussi depuis l'époque ou je communiquais pour la première fois cette vue à mon ami M. Sylvester, avais-je désesperé d'aller plus loin, l'application du théorème de M. Sturm n'étant pas praticable sur l'équation littérale et compliquée qui aurait la forme-type pour son premier membre.

La méthode suivante à laquelle je ne suis parvenu qu'après bien des efforts, me semble peut-être meriter un instant d'attention, car elle offrira si je ne me trompe, une étude algébrique complète des racines de l'équation générale du cinquième degré, sous la point de vue de la distinction de ces racines comme quantités réelles et imaginaires, lorsqu'on attribue aux coefficients toutes les valeurs réelles possibles. Je ferai précéder cette recherche de quelques lemmes, afin de ne pas interrompre par la suite l'ordre des raisonnements.

Lemmes Préliminaires.

Lemme 1^r^. Le produit des carrés des différences des racines d'une équation de degré quelconque $fx = 0$, est positif ou négatif, selon que le nombre des racines imaginaires de cette équation, est $\equiv 0$ ou $\equiv 2 \pmod 4$. Supposons cette proposition vraie pour une équation d'un degré déterminé $fx = 0$, nous allons démontrer qu'elle subsiste pour la nouvelle équation

$$Fx = (x - \alpha)(x - \beta) fx = 0.$$

Soit en effet D et D, les discriminants, ou pour plus de précision, les produits des carrés des différences des racines des équations $F = 0, f = 0$, on trouvera sans difficulté

$$D = (\alpha - \beta)^2 f^2(\alpha) f^2(\beta)\, \mathrm{D}.$$

D'ou l'on voit qu'en supposant réelles les racines α et β, D et D seront de même signe, tandis qu'en les supposant imaginaires conjuguées, D et D seront de signes contraires, car la produit $f(\alpha) f(\beta)$ sera positif, et le facteur $(\alpha - \beta)^2$ négatif. La proposition annoncée se vérifie donc à l'égard de l'équation $F = 0$ si elle a lieu pour l'équation $f = 0$; ainsi elle est générale, puisqu'elle est vraie dans le cas du second degré. Les exemples suivants montreront déjà un usage de cette remarque.

Considérons une forme biquadratique,

$$f = (a, b, c, b', a')(x, y)^4 = a(x - \alpha y)(x - \beta y)(x - \gamma y)(x - \delta y),$$

soit I l'invariant du second ordre

$$aa' - 4bb' + 3c^2,$$

et D le discriminant

$$a^6(\alpha-\beta)^2(\alpha-\gamma)^2 \ldots (\gamma-\delta)^2.$$

Je dis qu'en supposant $I<0$, la forme proposée aura deux ou quatre racines imaginaires, suivant que D sera négatif ou positif.

On a en effet, ce qui se vérifie très aisément,

$$I=aa'-4bb'+3c^2=\tfrac{8}{3}\{(\alpha-\beta)^2(\gamma-\delta)^2+(\alpha-\gamma)^2(\beta-\delta)^2+(\alpha-\delta)^2(\beta-\gamma)^2\},$$

donc l'hypothèse $I<0$, exclut le cas ou toutes les racines sont réelles, et le lemme précédent suffit pour distinguer l'un de l'autre les deux autres cas seuls possibles où le nombre des racines imaginaires est deux ou quatre. Quelque chose d'analogue a lieu aussi pour le cinquième degré, nous allons l'indiquer, bien que nous n'ayons pas à nous en servir par la suite. Soit,

$$f=(a,b,c,c',b',a')(x,y)^5=a(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y)(x-\varepsilon y),$$

D le discriminant, $a^8(\alpha-\beta)^2\ldots$ et Δ, l'invariant qui figure dans nos recherches, savoir,

$$(aa'-3bb'+2cc')^2-4(ab'-3bc'+3c^2)(a'b-3b'c+3c^2),$$

on trouvera,

$$\Delta=-\frac{7}{2^5}\Sigma(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\varepsilon)^2(\varepsilon-\alpha)^2,$$

la signe Σ se rapportant aux termes qu'on déduit de celui que nous avons écrit, par les permutations des racines.

Il s'en suit qu'on supposant Δ positif, la forme aura des racines imaginaires, et comme précédemment, elle en aura deux ou quatre, suivant que D sera négatif ou positif.

En passant remarquons encore cette relation,

$$D=a^8(\alpha-\beta)^2\ldots(\delta-\varepsilon)^2=5^5(\Delta^2+2^7J_3).$$

Lemme 2e. Il a été remarqué § 2, que les coefficients de la forme-type avaient pour commun dénominateur $(2J_3)^5$, d'après cela et pour plus de commodité nous considérons par la suite au lieu de ϕ, la forme $(2J_3)^5\phi$, *c. à. d.* nous ferons

$$\phi=(A,B,C,C',B',A')(\xi,\eta)^5,$$

les coefficients n'offrant plus J_3 en dénominateur, et ayant ainsi pour valeurs,

$$36A = \Delta^7 J_2 + \Delta^6 J_3 + 6\Delta^5 J_2^2 - 39\Delta^4 J_2 J_3 - 9\Delta^3(6J_3^2 - J_2^3) - 126\Delta^2 J_3 J_2^2 + 288\Delta J_3^2 J_2 + 1152 J_3^3,$$

$$36C = \Delta^6 J_2 + \Delta^5 J_3 + 6\Delta^4 J_2^2 - 27\Delta^3 J_2 J_3 - 3\Delta^2(14J_3^2 - 3J_2^3) - 90\Delta J_3 J_2^2 + 144 J_3^2 J_2,$$

$$36B' = \Delta^5 J_2 + \Delta^4 J_3 + 6\Delta^3 J_2^2 - 15\Delta^2 J_2 J_3 - 3\Delta(10J_3^2 - 3J_2^3) - 54 J_3 J_2^2 ;$$

$$9B = -I(\Delta^4 + 3\Delta^2 J_2 - 24\Delta J_3),$$

$$9C' = -I(\Delta^3 + 3\Delta J_2 - 12 J_3),$$

$$9A' = -I(\Delta^2 + 3J_2).$$

Cela posé, on aura ces relations remarquables,

$$(1)\quad \begin{cases} AB' - 4BC' + 3C^2 = 16\Delta J_3^5, \\ A'B - 4B'C + 3C'^2 = -16 J_3^5, \\ AA' - 3BB' + 2CC' = 0. \end{cases}$$

$$(2)\quad \begin{cases} A - 2\Delta C + \Delta^2 B' = 32 J_3^3, \\ B - 2\Delta C' + \Delta^2 A' = 0. \end{cases}$$

les premières résultent de l'expression du covariant quadratique de ϕ, qu'on obtient bien aisément. Effectivement, cette forme ϕ provient, par le fait de la suppression du dénominateur $(2J_3)^5$, de la transformée canonique F, par la substitution

$$\Delta(CX + C'Y) = 2J_3\xi, \quad \Delta\sqrt{\Delta}(CX - C'Y) = 2J_3\eta,$$

donc son covariant quadratique, proviendra par la même substitution du covariant $\sqrt{\Delta}XY$ rélatif à F, multiplié par la quatrième puissance du déterminant de la substitution, *c. à. d.* par $(2J_3)^4$. Cela donne pour le covariant quadratique de la forme-type, cette expression remarquable,

$$16 J_3^5(\Delta\xi^2 - \eta^2),$$

d'ou l'on tire de suite les équations (1).

En recherchant de la même manière le covariant linéaire du cinquième ordre de la forme-type, on obtiendra la valeur $10(2J_3)^{13}\xi$, mais d'après le loi générale de formation (§ 1e) ce covariant sera,

$$2^8 J_3^{10}\left(\Delta^2\frac{d^4\Phi}{d\eta^4} - 2\Delta\frac{d^4\Phi}{d\xi^2 d\eta^2} + \frac{d^4\Phi}{d\xi^4}\right) = 2^8 J_3^{10} 10\{(A - 2\Delta C + \Delta^2 B')\xi + (B - 2\Delta C' + \Delta^2 A')\eta\},$$

d'ou l'on conclut les rélations (2).

Il serait très important pour la théorie des formes du cinquième degré, de calculer comme nous venons de la faire, les valeurs d'un plus grand nombre de covariants de la forme-type, on recueillerait ainsi des éléments précieux d'observations qui pourraient éclairer la nature des rapports de ces covariants avec la forme dont ils tirent naissance. Pour le moment nous ne pouvons nous empêcher d'appeler l'attention du lecteur, sur la simplicité des équations que nous venons de trouver entre les coefficients si compliqués de la forme-type, elles vont nous donner une démonstration facile de la proposition suivante, qu'il importe d'établir pour la recherche spéciale que nous avons en vue dans ce §.

Lemme 3e. L'équation du 4e degré par rapport à J_2 qu'on forme en égalant à zéro l'invariant du 18e ordre, a toujours deux racines réelles, et deux racines imaginaires.

D'après la valeur que nous avons obtenue pour I^2, cette équation est

$$16I^2 = \Delta^5 J_2^2 + 2\Delta^4 J_2 J_3 + \Delta^3(J_3^2 + 6J_2^3) - 18\Delta^2 J_2^2 J_3 + 9\Delta(J_2^4 - 8J_2 J_3^2) - 24(2J_3^3 + 3J_2^3 J_3) = 0,$$

ou en ordonnant par rapport à J_2,

$$9\Delta J_2^4 + 6(\Delta^2 - 12J_3)J_2^3 + (\Delta^5 - 18\Delta^2 J_3)J_2^2 + (2\Delta^4 J_3 - 72\Delta J_3^2)J_2 + \Delta^3 J_3^2 - 48J_3^3 = 0.$$

Elle est comme on voit assez compliquée, pour qu'on puisse hésiter y appliquer le théorème de M. Sturm, mais heureusement elle admet une transformée très simple. Effectivement pour une valeur de J_2 qui satisfait à cette équation, les coefficients A, C, B', donnent en vertu des équations (1) et (2),

$$AB' + 3C^2 = 16\Delta J_3^5,$$

$$B'C = 4J_3^5,$$

$$A - 2\Delta C + \Delta^2 B' = 32J_3^3,$$

puisque B, C', A', contenant I en facteur, s'annullent. Or en éliminant A et B', on trouve

$$3C^4 - 8\Delta J_3^5 C^2 + 128J_3^8 C - 16\Delta^2 J_3^{10} = 0.$$

Ce résultat paraitra bien remarquable, si l'on a égard à la complication de la valeur de C exprimé en fonction de J_2; quoiqu'il en soit cette fonction étant rationnelle et entière par rapport à J_2, il suffira de raisonner sur l'équation en C, et d'établir qu'elle a bien deux racines réelles et deux

racines imaginaires. Or la premier point résulte de ce que le dernier terme est essentiellement négatif, et le second, de ce qu'en faisant

$$3C^4 - 8\Delta J_3^5 C^2 + 128 J_3^8 C - 16\Delta^2 J_3^{10} = (a, b, c, b', c', a')(C, 1)^4,$$

l'invariant du second ordre,

$$aa' - 4bb' + 3c^2,$$

a la valeur négative,

$$-\tfrac{128}{3}\Delta^2 J_3^{10}. \quad \text{(Lemme 1}^{\text{r}}\text{.)}$$

Des limites entre lesquelles se trouve toujours renfermé l'invariant J_2, de toute forme du cinquième degré à coefficients réels.

La forme-type a laquelle nous avons ramené la forme générale du cinquième degré par une substitution linéaire, ne contenant plus que trois paramètres, Δ, J_2, J_3, on est naturellement conduit à étudier les racines de cette forme considerées comme fonctions de ces paramètres, tandis qu'on n'aurait jamais songé à se proposer la même question sur les racines elles-mêmes de la forme primitive, considerées comme fonctions de cinq quantités arbitraires. Mais de l'abord de cette recherche, se présente une circonstance importante. En considérant pour les coefficients de la forme proposée, des valeurs réelles, les paramètres de la forme-type, qu'on ne devra pas déjà supposer imaginaires, ne peuvent même recevoir toutes les valeurs réelles possibles. Il entre en effet dans la forme-type, l'invariant I du 18^e ordre, qui doit être aussi essentiellement réel, de sorte que (étant algébriquement indépendantes) les quantités Δ, J_2, J_3, en tant qu'elles proviennent d'une forme réelle, sont assujeties à cette condition de rendre positive la fonction

$$16I^2 = 9\Delta J_2^4 + 6(\Delta^2 - 12J_3)\,J_2^3 + (\Delta^3 - 18\Delta^2 J_3)\,J_2^2 + (2\Delta^4 J_3 - 72\Delta J_3^2)\,J_2 + \Delta^3 J_3^2 - 48 J_3^3.$$

Or quels que soient Δ et J_3, nous avons démontré que l'équation $I^2 = 0$, en prenant J_2 pour inconnue, avait toujours deux racines réelles et deux racines imaginaires. Nommant donc j et j' ces racines réelles, il est aisé de voir qu'en supposant Δ positif, les valeurs de J_2 qui rendront la fonction I réelle, seront nécessairement au dehors de l'intervalle compris entre j et j', tandis qu'en supposant Δ négatif, ces valeurs seront comprises au contraire dans le même intervalle. Une observation très simple confirme cette conclusion que J_2 est nécessairement limité quand Δ est négatif; si l'on suppose en effet J_2 très grand, on trouvera en

employant les expressions données précédemment des coefficients A, B, etc., que la forme ϕ devient sensiblement proportionnelle à $(\xi\sqrt{\Delta}+\eta)^5$, de sorte que les cinq racines, se présenteraient toutes comme imaginaires, en devenant égales à la limite, tandis qu'on sait bien que leur commune valeur doit être réelle. Ces limites que nous venons de trouver pour les valeurs de J_2, vont encore se présenter dans une circonstance importante, comme on va voir.

Des limites entre lesquelles les racines de la forme-type sont des fonctions continues de J_2, considéré comme une variable réelle.

Nous nous fonderons pour cette recherche, sur ce théorème si important dans toute l'analyse, que l'illustre géomètre M. Cauchy a démontré sous un point de vue plus général dans les *Nouveaux Exercices de Mathématiques* (tom. II. p. 109), "Les racines d'une équation algébrique dont les coefficients contiennent sous forme rationnelle un paramètre, sont des fonctions continues de ce paramètre, tant qu'en variant suivant une loi donnée, en restant toujours réel par exemple, il n'atteint pas une des valeurs particulières qui font acquerir des racines égales à l'équation proposée. Mais la quantité J_2, que nous considérons comme un paramètre variable entrant dans l'équation que nous voulons etudier, savoir,

$$(A, B, C, C', B', A')(x, 1) = 0,$$

sous un radical carré I, nous ferons $y = Ix$, ce qui donnera l'équation en y,

$$(A, IB, I^2C, I^3C', I^4B', I^5A')(y, 1)^5 = 0,$$

dont tous les coefficients sont rationnels, puisque B, C, A' contiennent déjà I en facteur. Cela posé, nous allons pour appliquer la théorème de M. Cauchy, calculer son discriminant. Or le discriminant D, de la forme primitive

$$f = (a, b, c, c', b', a')(x, y)^5,$$

se reproduisant dans toute transformée, multiplié par la 20^e puissance du déterminant de la substitution, on trouvera d'abord $(2J_3)^{20}.D$ pour le discriminant de ϕ, et $(2J_3)^{20}.I^{20}.D$, pour celui de l'équation en y. Par là nous voyons que les valeurs de J_2 pour lesquelles les racines y deviennent discontinues, sont données par les équations

$$I = 0, \quad D = \Delta^3 + 2^7 J_2 = 0.$$

Et comme le radical carré I, est aussi fonction continue

de J_2, entre les limites déterminées par l'équation $I = 0$, la relation $y = Ix$, montre qu'on peut regarder les racines x elles-mêmes, comme fonctions continues de J_2, tant que cette variable, que nous supposons réelle, n'atteint pas la valeur $-2^{-7}\Delta^2$, ou l'une des quantités nommées précédemment j et j'. Peut-être devons nous faire observer, que nous ne considerons pas un autre genre de discontinuité, le passage à l'infini d'une racine, lorsque le coefficient A s'annulle. Le raison en est, que dans le voisinage d'une valeur réelle de J_2, qui donnerait $A = 0$, les inverses des cinq racines, sont certainement des fonctions continues, et ne pourront passer du réel à l'imaginaire, ou de l'imaginaire au réel, lorsque J_2 aura atteint et dépassé la valeur particulière en question. On voit donc qu'aucun changement dans la mode d'existence des racines de la forme-type, comme quantités réelles et imaginaires, ne correspond à cette discontinuité particulière qui provient du passage par l'infini, et qu'ainsi elle n'est pas à considerer dans notre recherche.*

Sur les valeurs des racines de la forme-type, lorsque J_2 est égal à la limite j ou à la limite j'.

Nous avons précédemment distingué avec soin, dans l'ensemble des valeurs réelles de J_2, les intervalles entre lesquels cette quantité peut être regardée comme provenant d'une forme à coefficients réels. Franchir les limites assignées, sera donc considerer ce que deviennent les racines de la forme-type, pour un état imaginaire des coefficients de la forme primitive. Cependant, si nous supposons toujours J_2 réel, ces valeurs imaginaires qui viendront nécessairement s'offrir, ne seront point entiérement arbitraires, et seront soumises à des conditions spéciales. Or on va voir combien est utile la considération de ces valeurs limitées comme nous le disons, de manière que les invariants du 4e, du 8e et du 12e ordre restent réels, l'invariant du 18e étant seul affecté du facteur $\sqrt{(-1)}$. Effectivement nous allons pouvoir suivre de la manière la plus facile et la plus claire, comment les racines de la forme-type changent successivement de

* Cette consideration des inverses des racines, sert aussi à établir, quand on recherche la distribution en systèmes circulaires des racines ν, d'une équation de la forme $N\nu^m + P\nu^{m-1} + \ldots = 0$, N, P, $\ldots$ étant des polynomes entiers en z, que ces systèmes subsistent sans altération, lorsque la contour décrit par la variable z, vient à comprendre un nombre quelconque de points, auxquels correspondent des racines de l'équation $N=0$. Voyez à ce sujet le N°. 37 du mémoire de M. Puiseux, intitulé 'Recherches sur les Fonctions Algébriques,' (*Journal de M. Liouville*, tom. xv.).

nature en passant du réel à l'imaginaire ou de l'imaginaire au réel, lorsque J_2 varie de $-\infty$ à $+\infty$, et par suite établir ce que sont ces racines dans une intervalle donné, résultat important, auquel nous n'aurions pu parvenir en renonçant à ces valeurs de paramètre variable qui supposent nécessairement imaginaires les coefficients de la forme proposée. Voici pour cet objet, les dernières propositions préliminaires que nous avons à démontrer. Je dis d'abord qu'en supposant $D = 0$, on aura

$$2^{32} I^2 = (25\Delta^3 - 3.2^{10} J_3)\,(25\Delta^3 - 2^{11} J_3)^2.$$

C'est une conséquence immédiate de la formule

$$\frac{I}{\sqrt{\Delta^3}} = ACB' - A'C'B^2 = \frac{48}{5} pq\,(2p^3 + q^3),$$

donnée § 5. On trouvera en effet le résultat annoncé en élevant au carré et remplaçant p^3 et q^3 par leurs valeurs en J_3 et Δ, telles qu'elles résultent des formules de ce §. Il s'ensuit que pour $D = 0$, I sera réel ou imaginaire, suivant la signe de la quantité $25\Delta^3 - 3.2^{10} J_3$, et par conséquent, le discriminant s'évanouira dans l'intervalle des valeurs admises ou des valeurs exclues de J_2, suivant que $25\Delta^3 - 3.2^{10} J_3$, sera positif ou négatif. Cela posé, je vais démontrer, que si la discriminant ne s'évanouit qu'en dehors des limites $J_2 = j$, $J_2 = j'$, les racines de la forme-type présenteront pour ces deux limites, un même nombre de quantités réelles et un même nombre de quantités imaginaires. Deux cas sont à distinguer suivant que Δ est positif ou négatif. Dans l'un et l'autre, les racines de la forme-type seront certainement entre les limites j et j' des fonctions continues de J_2, mais dans le premier il faut exclure les limites car en s'annullant le radical I passe alors brusquement de l'imaginaire au réel, et devient discontinue ; dans le second au contraire, les limites sont comprises, car le radical est réel avant de s'évanouir, et ne devient discontinue en passant à l'imaginaire, que si l'on franchit les limites j ou j'. C'est donc seulement pour ce second cas que notre proposition se trouve immédiatement établie, et sous ce point de vue, le premier exigerait une discussion que la méthode suivante évite, car il n'y figure plus de considérations de continuité.

Lorsque $I = 0$, nous avons trouvé précédemment les relations

(1) $AB' + 3C^2 = 16\Delta J_3^5$, $B'C = 4J_3^5$, $A - 2\Delta C + \Delta^2 B' = 32 J_3^3$,

et aussi une équation ne contenant que C, et que nous présenterons sous cette forme

$$(2)\quad (3C^2 + 4J_3^5\Delta)(C^2 - 4\Delta J_3^5) = -\,128 J_3^6 C.$$

Cela posé, il s'agit d'en déduire les valeurs des quantités qui déterminent par leurs signes la nature des racines de l'équation

$$(A, 0, C, 0, B', 0)(x, 1)^5 = 0.$$

Or ces quantités sont $25C^2 - 5AB'$, en premier lieu, puis les rapports $\frac{C}{A}$, $\frac{B'}{A}$, mais en leur place il sera préferable de prendre les suivantes $5C^2 - AB'$, AB', AC, ou même celles-ci

$$5C^2 - AB', \quad \frac{5C^2 - AB'}{AB'},$$

et $B'C$, ce qui est permis comme on le verra bien facilement. Mais par l'équation (1), on trouvera

$$5C^2 - AB' = 8(C^2 - 2\Delta J_3^5),$$

et

$$\frac{5C^2 - AB'}{AB'} = 8\,\frac{C^2 - 2\Delta J_3^5}{16\Delta J_3^5 - 3C^2},$$

ce qui nous conduit à déterminer la nature des racines de notre équation, par ces deux fonctions très simples,

$$u = C^2 - 2\Delta J_3^5, \quad v = \frac{C^2 - 2\Delta J_3^5}{16\Delta J_3^5 - 3C^2},$$

car il est inutile de considerer la troisième $B'C$, qui conserve absolument la même valeur pour $J_2 = j$, $J_2 = j'$.

Or en élevant au carré les deux membres de l'équation (2), on introduira partout le carré C^2, et une élimination facile alors, donnera

$$(3)\quad (3u + 10\Delta J_3^5)^2(u - 2\Delta J_3^5)^2 = 128^2 J_3^{16}(u + 2\Delta J_3^5),$$

$$(4)\quad \Delta^3(30v + 5)^2(2v - 1)^2 = 2.32^2 J_3(8v + 1)(3v + 1)^2.$$

Chacune de ces équations aura comme l'équation en C, deux racines imaginaires et deux racines réelles qui correspondent respectivement à $J_2 = j$, $J_2 = j'$, donc pour l'une et pour l'autre, les racines réelles seront de mêmes signes ou de signes contraires, suivant que le dernier terme sera positif ou négatif. Or le dernier terme de (3) est

$$(\tfrac{4}{3})^2\Delta^2 J_3^{20}(25\Delta^3 - 2^{11}J_3),$$

le dernier terme de (4),

$$12^2 \frac{25\Delta^3 - 3.2^{10}J_3}{25\Delta^3 - 2^{11}J_3},$$

et que Δ soit positif ou négatif, il est aisé de voir que ces quantités sont positives. En effet, pour $\Delta > 0$, elles le sont évidemment si J_3 est négatif, mais si J_3 est positif, la condition $25\Delta^3 - 3.2^{10}J_3 > 0$, qui est l'hypothèse, entraine $25\Delta^3 - 2^{11}J_3 > 0$, et notre proposition est vérifiée. Enfin pour $\Delta > 0$ elle est évidente si J_3 est positif, mais si J_3 est négatif, l'hypothèse qui est alors $25\Delta^3 - 3.2^{10}J_3 < 0$, entraine $25\Delta^3 - 2^{11}J_3 < 0$. Donc aux deux limites j et j', les trois quantités qui déterminent la nature des racines de la forme-type, ont individuellement les mêmes signes, et ces racines présentent dans ces deux cas un même nombre de quantités réelles et imaginaires.

Ce que deviennent successivement les racines de la forme-type lorsque J_2 varie de $-\infty$ à $+\infty$.

Nous distinguerons quatre cas principaux dans cette recherche, que nous traiterons dans l'ordre suivant:

Premier Cas : $\Delta > 0$, $25\Delta^3 - 3.2^{10}J_3 > 0$.

Second : $\Delta < 0$, $25\Delta^3 - 3.2^{10}J_3 > 0$.

Troisième : $\Delta > 0$, $25\Delta^3 - 3.2^{10}J_3 < 0$.

Quatrième : $\Delta < 0$, $25\Delta^3 - 3.2^{10}J_3 < 0$.

Premier Cas.

Les valeurs admises de J_2 forment alors deux séries, l'une de $-\infty$ à j, la seconde de j' à $+\infty$, (en nommant j la plus petite des quantités j et j') et la condition $25\Delta^3 - 3.2^{10}\Delta^3 > 0$, signifie comme il a été dit plus haut, que le discriminant s'évanouira necessairement, pour une valeur de J_2 comprise dans l'une des séries indiquées, nous admettrons pour fixer les idées que ce soit dans la première.

Cela posé, faisons croître J_2 par degrés insensibles à partir de $-\infty$; tant que le discriminant $D = \Delta^2 + 2^7 J_2$, ne viendra pas à s'annuller, les cinq racines resteront des fonctions continues, et aucun changement ne surviendra dans leur nature. Mais pour $J_2 = -2^{-7}\Delta^2$, deux d'entre elles et deux seulement deviendront égales, de sorte que dans le voisinage de cette valeur, elle pourront passer du réel à l'imaginaire

or de l'imaginaire au réel, en devenant discontinues, tandis que les trois autres resteront au contraire des fonctions continues de J_3.

En raison de cette circonstance essayons d'en déterminer la nature. Pour cela nous nous placerons précisément dans ce cas particulier ou $D = 0$. Divisant la forme-type par le facteur linéaire qu'elle contient alors au carré, nous obtiendrons une forme cubique, dont il faudra calculer le discriminant. Mais dans ce but, nous pouvons remplacer la forme-type Φ, par la transformée canonique F, puisqu'elle s'en déduit en faisant une substitution au déterminant réel $2J_3$. Alors un calcul très facile qui a été exécuté §5 (in finem) conduit abstraction faite d'un facteur positif à la fonction deja consideré plus haut

$$25\Delta^3 - 2^{11}J_3.^*$$

Il a été remarqué qu'elle était positive dans ce premier cas ou nous nous trouvons maintenant, ou l'on a les conditions

$$\Delta > 0, \quad 25\Delta^3 - 3.2^{10}J_3 > 0.$$

Ainsi de ces trois racines fonctions continues de J_2, entre les limites $J_2 = -\infty$, $J_2 = j$, une seule est réelle et les deux autres sont imaginaires. Cela posé, il s'agirait de reconnaitre pour des valeurs de J_2, infiniment voisines de $-2^{-7}\Delta^2$, la nature des deux autres racines qui sont égales pour $J_2 = -2^{-7}\Delta^2$. Cette question centre dans les principes connus, mais nous pouvons l'éviter en rapellant le premier lemme ou il a été établi que la seule condition $D > 0$ assurait l'existence de deux racines imaginaires et de trois racines réelles. Puisqu'il y a dans l'équation deux racines imaginaires quelque soit J_2, il faudra que les deux racines qui deviennent égales quand le discriminant s'évanouit, soient réelles, tant qu'il est négatif, et passent en devenant discontinues à l'imaginaire, lorsqu' après s'être annullé le discriminant devient positif. Maintenant J_2 continuant à croître, la forme-type offrira toujours quatre racines imaginaires et une racine réelle jusqu' à ce qu'on parvienne à la limite $J_2 = j$, à partir de laquelle on entre dans l'intervalle des valeurs exclues du paramètre. Alors les coefficients qui contiennent en facteur le radical carré deviennent dans tout cet intervalle, imaginaires, cependant nous allons encore suivre les racines en les faisant dépendre d'une équation

* J'ai pris suivant l'usage, le discriminant d'une forme cubique, de signe contraire au produit des carrés des différences des racines de cette forme.

à coefficients réels. Pour cela, faisans dans la proposée

$$(A, B, C, C', B', A')(x, 1)^5 = 0, \quad y = x\sqrt{(-1)},$$

nous aurons dans l'intervalle compris entre j et j', la transformée à coefficients réels

$$\{A, \sqrt{(-1)}, B, -C, -\sqrt{(-1)}\,C', B', \sqrt{(-1)}\,A'\}\,(y, 1)^5 = 0.$$

Dans cet intervalle, et les limites comprises les cinq racines y, seront fonctions continues de J_2, ainsi leur nature dépend de leurs valeurs initiales, par ex. pour $J_2 = j$. Mais il est bien à remarquer qu' alors, les quatre racines qui sont imaginaires, peuvent avoir leurs parties réelles nulles ; deux cas differents peuvent donc se présenter, les valeurs initiales des racines y, seront toutes réelles, ou bien quatre d'entre elles seront imaginaires et une seule réelle.

C'est une question curieuse et délicate, de reconnaitre si les deux cas sont possibles, ou lequel peut seulement avoir lieu. Pour le résoudre, je remarquerai que l'équation en y, pour $J_2 = j$ par ex. est de cette forme,

$$(A, 0, -C, 0, B', 0)(y, 1) = 0,$$

et que la quantité $\dfrac{B'}{A}$, est nécessairement positive. En effet si elle était négative, on voit bien aisément que cette équation aurait nécessairement deux racines imaginaires et très racines réelles. Et la même chose a lieu pour $J_2 = j'$, d'ou il suit que la signe commun aux deux racines réelles de l'équation en

$$\nu = \frac{C^2 - 2\Delta J_3^5}{16\Delta J_3^5 - 3C^2} = \frac{1}{8}\,\frac{5C^2 - AB'}{AB'},$$

sera celui de la quantité $5C^2 - AB'$, aux deux limites. Or l'équation en ν, a ses deux racines positives, car son premier membre, comme nous l'avons vu, est positif pour $\nu = 0$, et par la substitution ou la trouvera négatif au contraire pour $\nu = \frac{1}{2}$, donc nous avons une racine comprise entre zero et $\frac{1}{2}$, et l'autre racine qui est nécessairement de même signe, sera donc aussi positive. Etant ainsi assurés de signe des deux quantités $5C^2 - AB'$, considérons l'équation en u, qui a pour racines

$$u = \tfrac{1}{8}(5C^2 - AB') = C^2 - 2\Delta J_3^5.$$

Cette équation est

$$(3u + 10\Delta J_3^5)^2\,(u - 2\Delta J_3^5)^3 = (128)^2 J_3^{16}(u + 2\Delta J_3^5),$$

or ou faisant $u = \Delta J_3^5 U$, elle deviendra

$$\Delta^3(3U+10)^2(U-2)^2 = (128)^2 J_3(U+2),$$

ou $$\Delta^3(3U+10)^2(U-2)^2 - (128)^2 J_3(U+2) = 0.$$

Mais dans cette transformée, le premier membre est positif pour $U=0$, puisque par hypothèse on a

$$25\Delta^3 - 2^{11} J_3 > 0,$$

et pour $U=2$, il sera négatif. Donc les deux racines réelles et de même signe de cette transformée sont positives. Or pour qu'il en soit de même, comme nous l'avons trouvé d'ailleurs des deux racines $u = \Delta J_3^5 U$, la quantité J_3 doit être positive. C'est la donc une conséquence nécessaire de la supposition faite, que le discriminant s'évanouit entre les limites $J_2 = -\infty$, $J_2 = j$, et on en déduit par l'équation $B'C = 4J_3^5$ que la rapport $\frac{C}{B'}$ est positif. Donc enfin l'équation en y, pour $J_2 = j$ et $J_2 = j'$, a ses racines toutes réelles, et le premier des deux cas dont nous avions admis le possibilité a seul lieu.

Au de la de la limite $J_2 = j'$, les coefficients de l'équation en x, redeviennent réels, et dans cette seconde série des valeurs admises de paramètre, jusqu'à $J_2 = +\infty$, les cinq racines restent indéfiniment des fonctions continues, et offrent toujours une quantité réelle et quatre quantités imaginaires, dont les valuers initiales sont les produits du facteur $\sqrt{(-1)}$, par des quantités réelles.

Enfin considérons le cas ou le discriminant s'évanouit entre les limites $J_2 = +\infty$, $J_2 = j''$, et faisons alors de croître la paramètre variable, de $+\infty$ à $+\infty$. Tout-à-fait comme précédemment, nous trouverons dans l'intervalle compris entre les limites, $+\infty$ et j'', trois racines qui seront fonctions continues de J_2. Deux d'entre elles seront imaginaires et la troisième réelle, à cause de la condition $25\Delta^3 - 2^{11} J_3 > 0$. Quand aux deux autres qui deviennent égales quand le discriminant s'évanouit, elles seront imaginaires tant que le discriminant D restera positif, et passeront à l'état réel en devenant discontinues, lorsque D après s'être annulé deviendra négatif. Nous parvenons ainsi à la limite $J_2 = j''$, avec deux racines imaginaires et trois racines réelles. Pour suivre ultérieurement les racines, dans l'intervalle des valeurs exclues, de $J_2 = j''$ à $J_2 = j$, nous ferons encore $y = x\sqrt{(-1)}$ et la transformée à coefficients réels, aura dans toute cette étendue ses racines fonctions continues de J_2.

Quand a leur nature, elle résulte cette fois sans ambiguité des valeurs initiales, qui offrent trois quantités réelles, et deux quantités imaginaires, produits du facteur $\sqrt{(-1)}$, multiplié par des quantités réelles. Seulement nous observerons que J_3 doit être nécessairement dans ce cas, negatif; cela résulte très facilement de la discussion faite précédemment et nous pensons inutile de nous y arrêter. Enfin lorsque la paramètre décroit de la limite j à $-\infty$, nous rétrouvons pour les cinq racines des fonctions continues, parmi les quelles deux sont imaginaires et les trois autres réelles.

Second Cas.

Les valeurs admises de J_2 forment une seule série de j à j', et la condition

$$25\Delta^3 - 3.2^{10}J_3 > 0,$$

signifie que le discriminant s'évanouit dans cet intervalle. Faisant donc croître J_2 par degrés insensibles à partir de la limite j, tant qu'on n'atteindra pas la valeur $-2^{-7}\Delta^2$, pour la quelle D s'annule, les cinq racines demeureront des fonctions continues, et aucun changement ne surviendra dans leur nature. Mais pour $D = 0$, deux d'entre elles présenteront alors une discontinuité en devenant égales, tandis que les trois autres resteront des fonctions continues jusqu'à la limite j'. En raisonnant comme dans le cas précédent, on verra que la nature de ces trois racines dépend encore de l'expression $25\Delta^3 - 2^{11}J_3$, qui maintenant peut-être positive ou négative. Supposons le d'abord positive; c'est admettre dans l'intervalle compris entre j et j', l'existence de deux racines imaginaires et d'une racine réelle. Donc tant que le discriminant avant de s'évanouir restera négatif, les deux autres racines de l'équation seront réelles, est lorsque D deviendra positif après s'être annulé, elles passeront en devenant discontinues à l'état imaginaire. Ainsi donc dans ce cas, deux racines imaginaires et trois racines réelles à l'origine $J_2 = j$, et quatre racines imaginaires avec une racine réelle à la limite supérieure $J_2 = j'$. Maintenant si nous faisons encore $y = x\sqrt{(-1)}$, pour arriver à une transformée à coefficients réels entre les limites $J_2 = j$, $J_2 = -\infty$, d'une part, $J_2 = j'$, $J_2 = +\infty$, de l'autre, il est clair que dans ces deux intervalles les racines y, ne présenteront plus aucune discontinuité, et demeureront respectivement ce qu'elles sont aux deux origines. Or pour $J_2 = j$ nous savons avoir sur les cinq racines x trois quantités

réelles et deux imaginaires, donc il en sera de même pour les racines y. Et puisqu'il en est ainsi, l'expression $5C^2 - AB'$ est positive, alors nous en concluerons qu'elle sera négative pour $J_2 = j'$, car la dernier terme de l'équation en u, étant

$$(\tfrac{4}{3})^2 \Delta J_3^{20}(25\Delta^3 - 2^{11}J_3),$$

à cause de $\Delta < 0$, les deux racines u, sont de signes contraires. Donc les racines x présentant quatre quantités imaginaires pour $J_2 = j''$, il en sera de même des racines y.

Supposons en second lieu,

$$25\Delta^3 - 2^{11}J_3 < 0\,;$$

c'est admettre trois racines réelles comme fonctions continues de j à j''. Alors les deux autres racines qui sont égales quand D s'annulle, seront imaginaires pour $D < 0$, et deviendront réelles quand D passera à l'état positif. Ainsi comme tout-à-l'heure, deux racines imaginaires et trois racines réelles à l'origine, $J_2 = j$, mais cinq racines réelles à la limite $J_2 = j''$. Pour ce qui comme les quantités $y = x\sqrt{(-1)}$, de $J_2 = j$ à $J_2 = -\infty$, elles seront fonctions continues, et dans tout cet intervalle présenteront comme à l'origine, deux quantités imaginaires, et trois réelles. De $J_2 = j''$ à $J_2 = +\infty$, elles seront encore continues, mais une seule sera réelle, les quatre autres imaginaires, et ayant pour valeurs initiales les produits du facteur $\sqrt{(-1)}$, multiplié par des quantités réelles.

Troisième Cas.

Les deux derniers cas peuvent se ramener par la considération suivante au deux premiers.

Conçevons que dans la forme-type, en change Δ et J_3 en $-\Delta$ et $-J_3$, en conservant J_2 avec son signe, on vérifier que les coefficients A, B, C, C', B', A', deviendront respectivement $-A$, $B\sqrt{(-1)}$, C, $-C'\sqrt{(-1)}$, $-B'$, $A'\sqrt{(-1)}$; donc en mettant à la place de x, $x\sqrt{(-1)}$, et multipliant encore la transformée par $\sqrt{(-1)}$, on trouvera exactement le même résultat qu'en changeant les signes des invariants Δ et J_3. Or les conditions caractéristiques des deux derniers cas savoir,

$$\Delta > 0,\ 25\Delta^3 - 3.2^{10}J_3 < 0,\quad \text{et}\quad \Delta < 0,\ 25\Delta^3 - 3.2^{10}J_3 < 0,$$

reproduisant par le changement de signe de Δ et J_3, celles des deux premiers. Ainsi du second, nous allons déduire la troisième, et du premier la quatrième, avec ce seul

changement, que tout ce qui a été dit des quantités x et y, devra être transporté aux quantités y et x, x étant toujours l'inconnue de l'équation proposée, et y désignant $x\sqrt{(-1)}$. Cela donne les conclusions suivantes, en comment par le 3e cas. Alors les valeurs admises du paramètre, formant les deux séries de $-\infty$ à j et de j'' à $+\infty$.

Dans la première, des cinq racines x, deux sont imaginaires et trois réelles, dans la seconde, quatre sont imaginaires, une seule est réelle, et d'ailleurs dans les deux séries elles restent toutes fonctions continues du paramètre. Pour les racines y, c'est dans l'intervalle compris de j à j' qu'elle dépendent d'une équation à coefficients réels, et deux cas sont à distinguer suivant que $25\Delta^3 - 2^{11}J_3$ est négatif ou positif. Dans le premier, sur les trois racines qui sont fonctions continues de j à j', une est réelle, et deux sont imaginaires. Quand aux deux autres racines qui deviennent discontinues pour $D = 0$, elles sont réelles si D est négatif et imaginaires lorsque D est positif. Enfin si $25\Delta^3 - 2^{11}J_3$ est positif, les trois racines qui sont fonctions continues sont réelles, et les deux autres sont imaginaires pour $D < 0$ et réelles pour $D > 0$.

Quatrième Cas. Résumé.

En nous bornant pour abréger aux racines x, on voit qu'elles seront toutes fonctions continues du paramètres dans l'intervalle des valeurs admises, qui s'étend de j à j'. Maintenant et d'après ce qui a été dit du premier cas, toutes ces racines seront réelles si J_3 est positif, deux seront imaginaires et les trois autres réelles si J_3 est négatif. Ici on ne voit plus figurer le discriminant, cependant il est bien facile de vérifier encore que pour J_3 positif il a une valeur positive et pour J_3 négatif, une valeur négative. Effectivement, cette condition $J_3 > 0$ signifie comme nous l'avons demontré dans le premier cas, que le discriminant, s'évanouit entre les limites, $J_2 = -\infty$, $J_2 = j$, or pour des valeurs croissantes du paramètre, il passe en s'évanouissant, du négatif au positif, et arrive à l'état positif dans l'intervalle des valeurs admises. Au contraire si J_3 est négatif, il s'évanouit entre les limites $J_2 = j''$, $J_2 = +\infty$, et a conséquemment une valeur négative dans l'intervalle compris entre j et j'. Dans le 3e cas, le discriminant ne se trouve pas non plus immédiatement en évidence, mais comme il s'évanouit alors dans l'intervalle compris de j à j', il est clair qu'il est négatif de $J_2 = -\infty$ à $J_2 = j$, et positif de

$J_2 = j'$ à $J_2 = +\infty$. Ces remarques faites, nous pouvons maintenant rapprocher les divers résultats que nous venons d'obtenir ; nous formerons ainsi le tableau suivant, qui offre l'expression par les invariants fondamentaux, du nombre des racines réelles et imaginaires de l'équation génerale du cinquième degré.

$$\Delta^2 + 2^7 J_2 < 0 \ldots\ldots$$

trois racines réelles, deux racines imaginaires.

$$\Delta^2 + 2\,J_2^7 > 0 \begin{cases} \Delta < 0,\ 25\Delta^3 - 3.2^{10} J_3 < 0,\ J_3 > 0 \\ \qquad\qquad \text{Cinq racines réelles,} \\ \Delta < 0,\ 25\Delta^3 - 3.2^{10} J_3 > 0,\ 25\Delta^3 - 2^{10} J_3 < 0 \\ \qquad\qquad \text{Cinq racines réelles,} \\ \Delta > 0, \ldots \text{Une racine réelle, quatre imaginaires,} \\ \Delta < 0,\ 25\Delta^3 - 3.2^{10} J_3 > 0,\ 25\Delta^3 - 2^{11} J_3 > 0 \\ \qquad\qquad \text{Une racine réelle, quatre imaginaires.} \end{cases}$$

On comprend facilement, comment dans certaine cas la nombre des conditions a pu se réduire. Par exemple, avec $\Delta^2 + 2^7 J_2 > 0$ et $\Delta > 0$, on trouve une racine réelle et quatre racines imaginaires, lorsque $25\Delta^3 - 3.2^{10} J_3$ est positif, et aussi lorsqu'il est négatif, on peut donc ne conserver que les deux premières conditions. Enfin nous remarquerons, dans l'un des cas ou il y a cinq racines réelles, que les conditions $\Delta < 0$, $J_3 > 0$, entrainent la suivante, $25\Delta^3 - 3.2^{10} J_3 < 0$, qu'on pourra supprimer si l'on veut. La simplicité de ces résultats ne semble-t-elle pas indiquer que la théorème de M. Sturm, si beau dans sa généralité, est loin de fournir l'expression définitive, des conditions de réalité des racines des équations algébriques ?

Section VII.

Sur la réduite du 6e degré de l'équation générale du 5e degré.

Lagrange a fait voir que la résolution par radicaux de l'équation du 5e degré, dépend lorsqu'elle est possible, de la détermination d'une racine commensurable, d'une équation du 6e degré dont les coefficients dépendent rationnellement de ceux de la proposée. Mais jamais le calcul de cette réduite du 6e degré n'a été effective en général. La raison en est que les fonctions du cinq lettres les plus simples qui n'ont que six valeurs, étant au moins du second degré par rapport à l'une de ces lettres, les coefficients de

la réduite se présenteriont comme des fonctions des cinq coefficients de l'équation proposée, montant jusqu'au 12e degré, et contiendraient par suite plusieurs certaines de termes. Or on va voir qu'on peut vaincre cette difficulté à l'aide des résultats que nous avons obtenus sur les invariants des formes du 5e degré. Faisons en effet

$$f=(a,b,c,c',b',a')(x,y)^5=(x-\alpha y)(x-\beta y)(x-\gamma y)(x-\delta y)(x-\varepsilon y),$$

et considérons la fonction suivant des racines

$$J=a^4(\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\delta)^2(\delta-\varepsilon)^2(\varepsilon-\alpha)^2 + a^4(\alpha-\gamma)^2(\beta-\delta)^2(\gamma-\varepsilon)^2(\delta-\alpha)^2(\varepsilon-\beta)^2,$$

on reconnaitre bien facilement qu'elle est susceptible seulement de six valeurs, et en second lieu qu'elle est un invariant de la forme f. Il en résulte que les coefficients de l'équation du 6e degré en t, seront des fonctions rationnelles et entières de cas invariants fondamentaux, Δ, J_2, J_3, car l'invariant du 18e ordre n'y entrera pas, les degrés par rapport aux coefficients de f étant multiples de 4.

Ainsi qu'on représente cette équation en t par

$$t^6+(1)\,t^5+(2)\,t^4+(3)\,t^3+(4)\,t^2+(5)\,t+(6)=0,$$

(1), (2), etc. seront respectivement des fonctions linéaires des quantités plancées en regard, dans le tableau suivant:

(1) Δ

(2) Δ^2 J_2

(3) Δ^3 ΔJ_2 J_3

(4) Δ^4 $\Delta^2 J_2$ ΔJ_3 J_2^2

(5) Δ^5 $\Delta^3 J_2$ $\Delta^2 J_3$ ΔJ_2^2 $J_2 J_3$

(6) Δ^6 $\Delta^4 J_2$ $\Delta^3 J_3$ $\Delta^2 J_2^2$ $\Delta J_2 J_3$ J_2^3 J_3^2.

On voit donc qu'on est ainsi amené à un calcul relativement très facile, et que je me reserve de développer dans une autre occasion. J'exposerai alors les propriétés de cette équation en t, qui sont analogues à celles de l'équation modulaire pour la transformation du 5e ordre, sous ce point de vue que les fonctions non symétriques des racines qui s'expriment rationnellement par les coefficients, varient ou ne changent pas dans les deux cas par les mêmes permutations de ces racines. Cette équation en t est également intéressante en ce qu'elle offre la type d'une classe d'équations du 6e degré rédutibles au 5e. La propriété distinctive et caractéristique de cette classe d'équations, consiste en ce

que l'une des valeurs de ces fonctions des racines qui sans être symétrique par rapport à cinq d'entre elles n'ont cependant que six déterminations possibles, est alors nécessairement rationnelle.

Je ne terminerai pas ces recherches sur les formes du cinquième degré, sans rappeler que mon ami M. Sylvester, avait obtenu avant moi, dans son beau mémoire sur le calcul des formes, la notion des invariants du 4e, du 8e, et du 12e ordre. En donnant aux formes du cinquième degré, cette expression élégante,

$$ax^5 + by^5 + cz^5,$$

sous la condition $x + y + z = 0$, M. Sylvester a trouve pour ces invariants les valeurs

$$a^2b^2 + a^2c^2 + b^2c^2 - 2abc\,(a + b + c), \quad a^2b^2c^2(ab + ac + bc), \quad a^4b^4c^4,$$

qui sont des fonctions symétriques très simples des trois éléments a, b, c. Enfin l'invariant du 18e ordre qui joue un role si important dans ma théorie, s'est aussi présenté dans ses recherches, élevé au carré et indiquant lorsqu'il s'évanouit, l'impossibilité de la réduction à la forme citée,

$$ax^5 + by^5 + cz^5.$$

Exprimé en a, b, c, il a pour valeur

$$a^5b^5c^5\,(a - b)\,(a - c)\,(b - c),$$

expression encore bien simple, et qui montre sous des points de vues très différents comment on est conduit aux mêmes notions analytiques, dans cette vaste et féconde théorie des formes.

ON THE SEPARATION OF THE VARIABLES IN DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

By W. H. L. Russell.

The method of solving differential equations of the first order by separating the variables, was introduced soon after the invention of the infinitesimal calculus. John Bernoulli applied it to the integration of homogeneous equations. The process underwent great improvement in the hands of Euler. One of the equations he treated was the following:

$$ydx\,(c + nx) - dy\,(y + a + bx + nx^2) = 0,$$

in which the variables are separated by putting

$$y = \frac{z(a + bx + cx^2)}{c + nx - z}.$$

Since his time Jacobi has given the complete integral of the differential equation

$$(A + A'x + A''y)(x\,dy - y\,dx) - (B + B'x + B''y)\,dy + (C + C'x + C''y)\,dx = 0,$$

by a very elegant process which will be found in *Crelle's Journal*. I propose in the following paper to investigate more completely than has hitherto been done, a method by which we may ascertain how far such equations are integrable by the assumption

$$y = \frac{p_1 + p_2 z + p_3 z^2 + \dots}{q_1 + q_2 z + q_3 z^2 + \dots}$$

Let the differential equation be

$$U\frac{dy}{dx} = V, \text{ where } U \text{ and } V \text{ are functions of } (x) \text{ and } (y).$$

Let $y = \dfrac{p + qz}{r + sz}$, where p, q, r, s are functions of (x).

$$\text{Then } (r + sz)^2 \frac{dy}{dx} = (r + sz)\left\{\frac{dp}{dx} + \frac{dq}{dx}z + q\frac{dz}{dx}\right\} - (p + qz)\left\{\frac{dr}{dx} + \frac{ds}{dx}z + s\frac{dz}{dx}\right\}$$

$$= \left(r\frac{dp}{dx} - p\frac{dr}{dx}\right) + \left\{s\frac{dp}{dx} + r\frac{dq}{dx} - q\frac{dr}{dx} - p\frac{ds}{dx}\right\}z + \left(s\frac{dq}{dx} - q\frac{ds}{dx}\right)z^2 + (rq - sp)\frac{dz}{dx}.$$

Let $\dfrac{V}{U} = \dfrac{v}{u}$, where (v) and (u) are rational and entire functions of (x) and (z). Then the equation becomes

$$u(rq - sp)\frac{dz}{dx} = (r + sz)^2 v + \left(p\frac{dr}{dx} - r\frac{dp}{dx}\right)u + \left\{q\frac{dr}{dx} + p\frac{ds}{dx} - s\frac{dp}{dx} - r\frac{dq}{dx}\right\}uz + \left(q\frac{ds}{dx} - s\frac{dq}{dx}\right)uz^2.$$

We may expand (v) in terms of (z), ascending in powers of (z), and put

$$v = v_0 + v_1 z + v_2 z^2 + v_3 z^3 + \&c.$$

Now in the transformed equation the variables are to be separated. Hence it must be of the form $QXZ = QX'Z'$, where Q is a function of (x) and (z); X, X' are functions of (x) only, Z, Z' are functions of (z) only. Consequently we shall have

$$u = w(r + sz)^2 (c_0 + c_1 z + c_2 z^2 + \&c.) \ldots\ldots\ldots\ldots (I).$$

or $$u = w(r + sz)(c_0 + c_1 z + c_2 z^2 + \ldots) \ldots\ldots\ldots\ldots (II),$$

or else $$u = w(c_0 + c_1 z + c_2 z^2 + \ldots) \ldots\ldots\ldots\ldots\ldots (III),$$

where (w) is a function of (x) only. Let us take the form (I). Then the equation becomes

$$w(rq - sp)(c_0 + c_1 z + \ldots)\frac{dz}{dx} = v_0 + c_0\left(p\frac{dr}{dx} - r\frac{dp}{dx}\right)w$$

$$+ \left\{v_1 + wc_1\left(p\frac{dr}{dx} - r\frac{dp}{dx}\right) + w\left(q\frac{dr}{dx} + p\frac{ds}{dx} - s\frac{dp}{dx} - r\frac{dq}{dx}\right)\right\}z + \&c.$$

$= X_0 + X_1 z + X_2 z^2 + \ldots$ when X_0, X_1 ... are functions of (x). Hence the condition that the variables be separated in the second member of the equation in easily seen to be

$A_0 X_0 = A_1 X_1 = A_2 X_2 = \ldots$ when A_0, A_1 ... are any constants.

Since X_0, X_1, X_2 are rational and entire functions of (x), we may put

$$X_0 = a_0 + b_0 x + c_0 x^2 + \ldots, \quad X_1 = a_1 + b_1 x + c_1 x^2 + \&c.,$$
$$X_2 = a_2 + b_2 x + c_2 x^2 + \ldots$$

Hence we see that the conditions of separation are

$$\frac{b_0}{a_0} = \frac{b_1}{a_1} = \frac{b_2}{a_2} = \ldots \qquad \frac{c_0}{a_0} = \frac{c_1}{a_1} = \frac{c_2}{a_2} = \ldots$$

I need scarcely remark that the variables are already separated in the first member of the equation. Hence these equations contain the solution of the Problem, under the condition implied by equation (I). In cases (II) and (III) the investigation will be exactly analogous.

I shall now proceed to some examples. For the present I shall assume p, q, r, s to be of the first degree and to be given by the following equations:

$$p = a + bx, \quad q = a' + b'x, \quad r = m + nx, \quad s = m' + n'x.$$

Let the differential equation

$$(rq - sp)^2 \frac{dy}{dx} = (\mu + \phi x)\, q^2 - (\mu' + \phi' x)\, pq + (\mu'' + \phi'' x)\, p^2$$
$$- \{2(\mu + \phi x)\, qs - (\mu' + \phi' x)(rq + sp) + 2(\mu'' + \phi'' x)\, rp\}\, y$$
$$+ \{(\mu + \phi x)\, s^2 - (\mu' + \phi' x)\, rs + (\mu'' + \phi'' x)\, r^2\}\, y^2.$$

Then the variables are separated by assuming $y = \frac{p + qz}{r + sz}$, provided that

$$\frac{\mu + na - mb}{\phi} = \frac{\mu' + na' + n'a - m'b - mb'}{\phi'} = \frac{\mu'' + a'n' - m'b'}{\phi''},$$

and the transformed equation is

$$\phi(rq - sp)\frac{dz}{dx} = (\mu + na - mb + \phi x)(\phi + \phi' z + \phi'' z^2).$$

Let the differential equation be

$$(q - sy)(rq - sp)\frac{dy}{dx} = \mu q^2 - \mu' pq + \mu'' p^2$$
$$- \{2\mu qs - \mu'(rq + sp) + 2\mu'' rp\}\, y + \{\mu s^2 - \mu' rs + \mu'' r^2\}\, y^2.$$

Then the variables are separated by the same assumption as before, provided that

$$\frac{\mu m + na - mb}{\mu n} = \frac{\mu m' + \mu' m + a'n + n'a - bm' - mb'}{\mu n' + \mu' n}$$
$$= \frac{\mu' m' + \mu'' m + a'n' - b'm'}{\mu n' + \mu'' n} = \frac{m'}{n'},$$

and the transformed equation is

$$\mu n(rq - sp)\frac{dz}{dx} = (\mu m + na - mb + \mu n x)$$
$$\{\mu n + (\mu n' + \mu' n) z + (\mu' n' + \mu'' n) z^2 + \mu'' n z^3\}.$$

As instances of this transformation, I take the following particular cases of the equation which was mentioned at the commencement of this paper as treated by Jacobi.

$$(1 + xy)\frac{dy}{dx} = 1 - y + y^2.$$

Let $y = \frac{1 + x + z}{1 - xz}$, and we have

$$(1 + x + x^2)\frac{dz}{dx} = -xz(1 + z + z^2).$$

Again, let $(1+xy)\dfrac{dy}{dx} = 2 - 2y + y^2$.

Let $y = \dfrac{1+2x-z}{1+x+xz}$, and we have

$$(1+2x+2x^2)\frac{dz}{dx} = -x(1+z)(1+z^2).$$

Let the equation be

$$x(q-sy)(rq-sp)\frac{dy}{dx} = (\mu+\phi x)q^2 - (\mu'+\phi' x)pq + (\mu''+\phi'' x)p^2$$
$$-\{2(\mu+\phi x)qs - (\mu'+\phi' x)(rq+sp) + 2(\mu''+\phi'' x)rp\}y$$
$$+\{(\mu+\phi x)s^2 - (\mu'+\phi' x)rs + (\mu''+\phi'' x)r^2\}y^2.$$

Then if

$$\frac{\mu n+\phi m+an-bm}{m\mu} = \frac{m'\phi+n'\mu+m\phi'+n\mu'+a'n+an'-m'b-mb'}{m'\mu+m\mu'}$$
$$= \frac{m'\phi'+n'\mu'+m\phi''+n\mu''+n'a'-m'b'}{m'\mu'+m\mu''} = \frac{m'\phi''+\mu''n'}{m'\mu''},$$

and $$\frac{n\phi}{m\mu} = \frac{n'\phi+n\phi'}{m'\mu+m\mu'} = \frac{n'\phi'+n\phi''}{m'\mu'+m\mu''} = \frac{n'\phi''}{m'\mu''},$$

the transformed equation will be

$$m\mu x(rq-sp)\frac{dz}{dx} = \{m\mu+(n\mu+\phi m+an-bm)x+n\phi x^2\}$$
$$\{m\mu+(m'\mu+m\mu')z+(m'\mu'+m\mu'')z^2+m'\mu''z^3\}.$$

Let the equation be

$$(rq-sp)(ry-p)\frac{dy}{dx} = \mu q^2 - \mu' pq + \mu'' p^2$$
$$-\{2\mu qs - \mu'(rq+sp) + 2\mu'' rp\}y + (\mu s^2 - \mu' rs + \mu'' r^2)y^2.$$

Then if $y = \dfrac{p+qz}{r+sz}$, and

$$\frac{m}{n} = \frac{m\mu'+m'\mu+na-mb}{\mu' n+\mu n'} = \frac{m\mu'+m\mu''+na'+n'a-bm'-b'm}{n'\mu'+n\mu''}$$
$$= \frac{m'\mu''+n'a'-m'b'}{\mu''n'},$$

the transformed equation will be

$$\mu nz(rq - sp)\frac{dz}{dx} = \{\mu(m + nx)\}$$
$$\{\mu n + (\mu' n + \mu n')z + (n'\mu' + n\mu'')z^2 + \mu'' n' z^3\}.$$

Let

$$x(rq - sp)(ry - p)\frac{dy}{dx} = (\mu + \phi x)q^2 - (\mu' + \phi' x)qp + (\mu'' + \phi'' x)p^2$$
$$- \{2(\mu + \phi x)qs - (\mu' + \phi' x)(rq + sp) + 2(\mu'' + \phi'' x)rp\}y$$
$$+ \{(\mu + \phi x)s^2 - (\mu' + \phi' x)rs + (\mu'' + \phi'' x)r^2\}y^2.$$

Then, if $\dfrac{m\mu}{n\phi} = \dfrac{m\mu' + m'\mu}{n\phi' + n'\phi} = \dfrac{m\mu'' + \mu' m'}{n\phi'' + \phi' n} = \dfrac{\mu'' m'}{\phi'' n'}$,

$$\frac{\mu n + \phi m}{n\phi} = \frac{n\mu' + m\phi' + m'\phi + n'\mu + na - mb}{n\phi' + n'\phi}$$
$$= \frac{\mu'' n + \phi'' m + \phi' m' + n'\mu' + a'n + an' - m'b - mb'}{n\phi'' + n'\phi'}$$
$$= \frac{\mu'' n' + \phi'' m' + n'\phi' - b'm'}{\phi'' n'},$$

the transformed equation will be

$$n\phi\, xz(rq - sp)\frac{dz}{dx} = (m + nx)(\mu + \phi x)$$
$$\{n\phi + (n\phi' + \phi n')z + (n\phi'' + \phi' n')z^2 + \phi'' n' z^3\}.$$

Let us now assume the quantities p, q, r, s, to rise above the first degree, and let $p = a + bx + cx^2$, $q = a' + b'x + c'x^2$, $r = m + nx + hx^2$, $s = m' + n'x + h'x^2$), and let the equation be

$$(q - sy)(rq - sp)\frac{dy}{dx} = (\mu + \phi x)q^2 - (\mu' + \phi' x)pq + (\mu'' + \phi'' x)p^2$$
$$- \{2(\mu + \phi x)qs - (\mu' + \phi' x)(rq + sp) + (\mu'' + \phi'' x)rp\}y$$
$$+ \{(\mu + \phi x)s^2 - (\mu' + \phi' x)rs + (\mu'' + \phi'' x)r^2\}y^2.$$

Then, if

$$\frac{\mu m + an - bm}{\phi h} = \frac{\mu' m + \mu m' + a'n + an' - m'b - mb'}{\phi' h + \phi h'}$$
$$= \frac{a'n' - b'm' + \mu' m' + \mu'' m}{\phi' h' + \phi'' h} = \frac{\mu'' m'}{\phi'' h'},$$

$$\frac{m\phi+\mu n+2ah-2cm}{\phi h} = \frac{\phi' m+\mu' n+\phi m'+\mu n'+2a'h+2ah'-2cm'-2c'm}{\phi' h+\phi h'}$$

$$= \frac{2a'h'-2c'm'+\mu' n'+m'\phi'+\phi'' m+\mu'' n}{\phi' h'+\phi'' h} = \frac{\phi'' m'+\mu'' n'}{\phi'' h'},$$

$$\frac{\mu h+n\phi+bh-cn}{\phi h} = \frac{\phi' n+\mu' h+\phi n'+\mu h'+b'h'+bh'-c'n-cn'}{\phi' h'+\phi'' h}$$

$$= \frac{b'h'-c'n'+\phi' n'+h'\mu'+\phi'' n+h\mu''}{\phi' h'+\phi'' h} = \frac{\phi'' n'+h'\mu''}{\phi'' h'},$$

the transformed equation will be

$$\phi h(rq-sp)\frac{dz}{dx} = \{(\mu n+na-mb)$$

$$+(n\mu+m\phi+2ah-2cm)x+(\mu h+n\phi+bh-cn)x^2+\phi hx^3\}$$

$$\{\phi h+(\phi' h+\phi h')z+(\phi' h'+\phi'' h)z^2+\phi'' h'z^3\}.$$

Let the equation be

$$(q-sy)\frac{dy}{dx} = \{(\mu+\phi x)q-(\mu'+\phi' x)p\}-\{(\mu+\phi x)s-(\mu'+\phi' x)r\}y.$$

Then, if $r = m+nx, \quad s = m'+n'x,$

$$\frac{m^2\mu+na-bm}{n^2\phi} = \frac{2mm'\mu+m^2\mu'+a'n-n'a-m'b-mb'}{n^2\phi'+2nn'\phi}$$

$$= \frac{\mu m'^2+2mm'\mu'+a'n'-b'm'}{\phi n'^2+2nn'\phi'} = \frac{m'^2\mu'}{n'^2\phi'},$$

$$\frac{m^2\phi+2mn\mu-2cm}{n^2\nu} = \frac{2mm'\phi+2m'\mu n+2\mu mn'+m^2\phi'+2mn\mu'-2c'm-2cm'}{\phi n'^2+2nn'\phi'}$$

$$= \frac{2m'n'\mu+\phi m'^2+2mm'\phi'+2m\mu'n'+2nm'\mu'-2c'm}{\phi' n'^2+2nn'\phi'} = \frac{2m'n'\mu'+n'm'^2}{n'^2\phi'},$$

$$\frac{\mu n^2+2mn\phi-nc}{n^2\phi} = \frac{2nn'\mu+2n\phi m'+2n'\phi m-c'n-cn'+n^2\mu'+2mn\phi'}{\phi' n^2+2nn'\phi}$$

$$= \frac{\mu n'^2+2m'n\phi+2nn'\mu'+2n\phi' m'+2n'\phi' m-c'n'}{\phi' n'^2+2nn'\phi'} = \frac{n'^2\mu'+2nn'\nu}{n'^2\phi'},$$

the transformed equation will be

$$n^2\phi(rq-sp)\frac{dz}{dx} = \{m^2\mu+na-bm$$

$$+(m^2\phi+2mn\mu-2cm)x+(\mu n^2+2mn\phi-nc)x^2+n^2\phi x^3\}$$

$$\{n^2\phi+(\phi' n^2+2nn'\phi)z+(\phi' n'^2+2nn'\phi')z^2+n'^2\phi' z^3\}.$$

Let

$$x^2(rq-sp)^2\frac{dy}{dx}=(\mu+\phi x^4)q^2-(\mu'+\phi'x^4)qp+(\mu''+\phi''x^4)p^2$$
$$-\{2(\mu+\phi x^4)qs-(\mu'+\phi'x^4)(rq+sp)+2(\mu''+\phi''x^4)rp\}y$$
$$+\{(\mu+\phi x^4)s^2-(\mu'+\phi'x^4)rs+(\mu''+\phi''x^4)r^2\}y^2.$$

Then, if

$$\frac{na-mb}{\mu}=\frac{na'+n'a-bm'-b'm}{\mu}=\frac{a'n'-b'm'}{\mu''},$$

$$\frac{ah-cm}{\mu}=\frac{a'h+h'a-cm'-c'm}{\mu'}=\frac{a'h'-c'm'}{\mu''},$$

$$\frac{\phi+bh-nc}{\mu}=\frac{\phi'+b'h-c'n+bh'-cn'}{\mu'}=\frac{\phi''+b'h'-c'n'}{\mu''},$$

the transformed equation will be

$$\mu x^2(rq-sp)\frac{dz}{dx}=\{\mu+(an-bm)x^2+2(ah-cm)x^3$$
$$+(\phi+bh-nc)x^4\}\{\mu+\mu'z+\mu''z^2\}.$$

Let $(rq-sp)(ry-p)^2\frac{dy}{dx}=\mu(q-sy)^2+\mu'(ry-p)^2.$

Then, if

$$\frac{m}{n}=\frac{m'}{n'}=\frac{an-bm}{2(ah-cm)}=\frac{m\mu'+a'n+an'-m'b-bm'}{\mu'n+2(a'h+ah'-m'c-c'm)}$$
$$=\frac{m'\mu'+a'n'-b'm'}{n'\mu'+2(a'h'-c'm')},$$

$$\frac{m}{h}=\frac{m'}{h'}=\frac{an-bm}{bh-cn}=\frac{m\mu'+a'n+an'-m'b-bm'}{\mu'h+b'h+bh'-cn'-c'n}$$
$$=\frac{m'\mu'+a'n'-b'm'}{\mu'h'+b'h'-c'n'},$$

the transformed equation will be

$$\mu mz^2(rq-sp)\frac{dz}{dx}=\mu(m+nx+hx^2)$$
$$\{m\mu+m'\mu z+(an-bm)z^2+(m\mu'+a'n+an'-m'b-mb')z^3$$
$$+(m'\mu'+a'n'-b'm')z^4\}.$$

Let now the differential equation be

$$x^2(q-sy)(rq-sp)\frac{dz}{dx} = (\mu+\phi x^3)q^2 - (\mu+\phi x^3)qp + (\mu+\phi x^3)p^2$$
$$-\{2(\mu+\phi x^3)qs - (\mu'+\phi' x^3)(rq+sp) + 2(\mu''+\phi'' x^3)rp\}y$$
$$+\{(\mu+\phi x^3)s^2 - (\mu'+\phi' x^3)rs + (\mu''+\phi'' x^3)r^2\}y^2.$$

Then, if $\dfrac{n}{m} = \dfrac{n\mu'+n'\mu}{m\mu'+m'\mu} = \dfrac{n\mu''+n'\mu'}{m\mu''+m'\mu'} = \dfrac{n'}{\mu''}$,

$$na - mb = a'n + an' - m'b - mb' = a'n' - m'b' = 0,$$

$$\frac{m\phi-2cm}{m\mu} = \frac{m\phi'+m'\phi-2cm'-2c'm}{m\mu'+m'\mu} = \frac{m\phi''+m'\phi'-2c'm'}{m\mu''+m'\mu'} = \frac{m'\phi''}{\mu''m'}$$
$$-\frac{cn}{m\mu} = \frac{n\phi'+n'\phi-cn'-c'n}{m\mu'+m'\mu} = \frac{n\phi''+n'\phi'-c'n'}{m\mu''+m'\mu'} = \frac{\phi''n'}{\mu''m'}.$$

Then, if $y = \dfrac{p+qz}{r+sz} = \dfrac{a+bx+cx^2+(a'+b'x+c'x^2)z}{m+nx+(m'+n'x)z}$,

the transformed equation will be

$$m\mu x^2(rq-sp)\frac{dz}{dx} = \{m\mu+n\mu x+(n\phi-2cm)x^3-cnx^4\}$$
$$\{m\mu+(m\mu'+\mu m')z+(m\mu''+m'\mu')z^2+\mu''m'z^3\}.$$

I now proceed to investigate the criterion of integrability when we assume

$$y = \frac{p_1+p_2z+p_3z^2}{r_1+r_2z+r_3z^2}.$$

Let $U\dfrac{dy}{dx} = V$, as before, be the given equation; and let $\dfrac{V}{U} = \dfrac{v}{u}$, where (v) and (u) are rational and entire functions of (x) and (z). The transformed equation will be

$$u\{(r_1p_2-p_1r_2)+2(r_1p_3-p_1r_3)z+(p_3r_2-r_3p_2)z^2\}\frac{dz}{dx}$$
$$= (r_1+r_2z+r_3z^2)^2 v$$
$$+\left\{p_1\frac{dr_1}{dx}-r_1\frac{dp_1}{dx}\right\}u+\left\{p_1\frac{dr_2}{dx}+p_2\frac{dr_1}{dx}-r_1\frac{dp_2}{dx}-r_2\frac{dp_1}{dx}\right\}uz$$
$$+\left\{p_2\frac{dr_2}{dx}+p_1\frac{dr_3}{dx}+p_3\frac{dr_1}{dx}-r_2\frac{dp_2}{dx}-r_3\frac{dp_1}{dx}-r_1\frac{dp_3}{dx}\right\}uz^2$$
$$+\left\{p_2\frac{dr_3}{dx}+p_3\frac{dr_2}{dx}-r_2\frac{dp_3}{dx}-r_3\frac{dp_2}{dx}\right\}uz^3+\left\{p_3\frac{dr_3}{dx}-r_3\frac{dp_3}{dx}\right\}uz^4.$$

Let $v = v_0 + v_1 z + v_2 z^2 + \ldots$, when v_0, v_1, v_2 are functions of (x), $u = u_0 + u_1 z + u_2 z^2 + \ldots$

Then, if we substitute these in the given equation, we may write it thus:

$$(x_0 + x_1 z + x_2 z^2 + \ldots)\frac{dz}{dx} = (x_0' + x_1' z + x_2' z^2 + \ldots).$$

Then, in order that the variables in this equation may be separated, we put

$$Ax_0 = Bx_1 = Cx_2 = \&c.,$$
$$A'x_0' = B'x_1' = C'x_2' = \&c.,$$

where A, B, &c., A', B', &c. are any constants. Then, by equating like powers of (x) in these equations, we obtain the required conditions of integrability.

Let the differential equation be

$$(r_1 + r_2\rho + r_3\rho^2)^2\frac{dy}{dx} = (\mu + \phi x) + (\mu' + \phi' x)\rho + (\mu'' + \phi'' x)\rho^2 + (\mu^{(3)} + \phi^{(3)} x)\rho^3 + (\mu^{(4)} + \phi^{(4)} x)\rho^4,$$

where (ρ) is a root of the equation

$$\rho^2(r_3 y - p_3) + \rho(r_2 y - p_2) + (r_1 y - p_1) = 0,$$

and $\quad p_1 = a + bx, \quad p_2 = a' + b'x, \quad p_3 = a'' + b''x,$

$$r_1 = m + nx, \quad r_2 = m' + n'x, \quad r_3 = m'' + n''x.$$

Then the conditions that the variables be separated in the transformed equation will be

$$\frac{a'n + b'm - bm' - an'}{a'm - am'} = \frac{a''n + b''m - an'' - m''b}{ma'' - m''a} = \frac{b''m' + a''n' - a'n'' - b'm''}{a''m' - a'm''},$$

$$\frac{b'n - n'b}{a'm - am'} = \frac{b''n - n''b}{ma'' - m''a} = \frac{b''n' - n''b'}{a''m' - a'm''}.$$

And $$\frac{\mu + na - mb}{\phi} = \frac{\mu' + n'a - m'b + na' - mb'}{\phi'}$$

$$= \frac{\mu'' + n'a' - m'b' + n''a - m''b + na'' - mb''}{\phi''}$$

$$= \frac{\mu^{(3)} + a'n'' - m'b'' + a''n' - m''b'}{\phi^{(3)}} = \frac{\mu^{(4)} + n''a'' - m''b''}{\phi^{(4)}}.$$

Let the differential equation be

$$(r_1 + r_2\rho + r_3\rho^2)\frac{dy}{dx} = \mu + \phi x + (\mu' + \phi' x)\rho + (\mu'' + \phi'' x)\rho^2,$$

where ρ, r_1, p_1, &c., are the same as in the last example.

Then the conditions that the variables be separated in the transformed equation are as follow:

$$\frac{\mu m + na - mb}{\phi n} = \frac{\mu m' + \mu' m + an' + a'n - mb' - m'b}{\phi n' + \phi' n}$$

$$= \frac{\mu m'' + \mu' m' + m\mu'' + an'' - m''b + a''n - mb'' + n'a' - m'b'}{\phi n'' + \phi' n' + n\phi''}$$

$$= \frac{\mu' m'' + m'\mu'' + a'n'' + a''n' - m'b'' - m''b'}{\phi' n'' + \phi'' n'} = \frac{\mu'' m'' + a''n'' - m''b''}{\phi'' n''},$$

$$\frac{\phi m + \mu n}{\phi n} = \frac{\phi m' + n'\mu + m\phi' + \mu' n}{\phi n' + \phi' n}$$

$$= \frac{m''\phi + n''\mu + \phi' m' + \mu' n + n\mu'' + \phi'' m}{\phi n'' + \phi' n' + n\phi''}$$

$$= \frac{\phi' m'' + \mu'\phi'' + n'\mu'' + m'\phi''}{\phi' n'' + n'\phi''} = \frac{\phi'' m'' + n''\mu''}{\phi'' n''}.$$

REMARKS ON INTEGRATION.

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It is proposed, in the following paper, to offer some practical remarks upon the subject of integration, more especially in reference to three capital defects under which many of the methods in common use appear to me to labour. The first defect in the methods, to which allusion is made, is their extremely artificial character, which occasions much embarrassment to the student at first, and considerable difficulty in his effort to retain them. The second great defect in these methods is, that they seem wholly unsusceptible of useful generalization. The third defect is less common, and consists in this, that some of the processes employed are circuitous, terms being introduced which subsequent operations cause to disappear. The method here put forward appears to be free from these defects, and, in so far as it is calculated to reduce and simplify the labours of the student, a practical good.

In illustration, I shall apply the method to the well-known equations

$$xD^2y + Dy + y = 0,$$

(Gregory's *Examples*, chap. v., Ex. 4,)

and $$D^2y + \frac{2}{x}Dy + \left(n^2 - \frac{2}{x^2}\right)y = 0,$$

(Gregory's *Examples*, chap. IV., sect. 2, Ex. 10.)

In quoting from the manual just named, I would desire to express a sense of its general merit as well as personal obligation to its study.

If we multiply the first equation by x and the second by x^2, they become respectively

$$(xD)^2 . y + xy = 0,$$

$$(xD - 1)(xD + 2) . y + n^2x^2y = 0.$$

Now the common type of these equations is

$$F(xD)y + Mx^my = 0;$$

or, more generally,

$$F(xD)y + Mx^my = X.$$

Let us suppose that

$$X = \Sigma A_a x^a,$$

and proceed to solve the more general type.

Operating on both sides with the inverse of $F(xD)$, we get

$$y + \frac{1}{F(xD)} Mx^m . y = \frac{1}{F(xD)} X + \frac{1}{F(xD)} 0,$$

or

$$y + \frac{1}{F(xD)} Mx^m . y = \Sigma \frac{A_a x^a}{F(a)} + \Sigma C_a x^a,$$

where the last term is the ordinary complementary function upon the supposition that all the roots of

$$F(u) = 0$$

are real and unequal, and in which, if any modification should arise from the existence of equal or imaginary roots, the generality of the method is not affected.

Now, dissecting the operator in the left-hand member from its subject, and operating with the expansion of its inverse upon the right-hand member, we get

$$y = \begin{cases} \left\{1 - \frac{1}{F(xD)} Mx^m + \frac{1}{F(xD)} Mx^m . \frac{1}{F(xD)} Mx^m - \&c.\right\} . \Sigma \frac{A_a x^a}{F(a)} \\ + \\ \left\{1 - \frac{1}{F(xD)} Mx^m + \frac{1}{F(xD)} Mx^m . \frac{1}{F(xD)} Mx^m - \&c.\right\} . \Sigma C_a x^a, \end{cases}$$

or the required solution is at once

$$y = \begin{cases} \Sigma \dfrac{A_\alpha x^\alpha}{F(a)} \left\{1 - \dfrac{Mx^m}{F(a+m)} + \dfrac{(Mx^m)^2}{F(a+2m)\,F(a+m)} - \&c.\right\} \\ + \\ \Sigma C_\alpha x^\alpha \left\{1 - \dfrac{Mx^m}{F(\alpha+m)} + \dfrac{(Mx^m)^2}{F(\alpha+2m)\,F(\alpha+m)} - \&c.\right\}, \end{cases}$$

the coefficients within brackets in the first and second great terms differing merely in the substitution of α for a.

In this method no mathematical artifice is employed, and the result seems to be obtained in the most direct manner. That the method admits of easy generalization can be readily now shown.

Let the partial differential equation to be solved be represented by the type

$$F(\nabla)z + \Theta_m z = \Omega,$$

where Θ_m is an homogeneous function in x and y of the m^{th} degree, Ω is a mixed function of x, y, and ∇ is the index-symbol

$$xD_x + yD_y.$$

Operate with

$$\frac{1}{F(\nabla)},$$

having broken up Ω into sets of homogeneous functions; there results

$$\left(1 + \frac{1}{F(\nabla)}\Theta_m\right) z = \Sigma \frac{\Theta_\alpha}{F(a)} + \Sigma u_\alpha,$$

where u_α is a homogeneous function of the given degree α, but arbitrary in form, and operating on both sides with the inverse of

$$\left(1 + \frac{1}{F(\nabla)}\Theta_m\right),$$

we get at once, as before,

$$z = \begin{cases} \Sigma \dfrac{\Theta_\alpha}{F(a)} \left\{1 - \dfrac{\Theta_m}{F(a+m)} + \dfrac{\Theta_m^{\,2}}{F(a+2m)\,F(a+m)} - \&c.\right\} \\ + \\ \Sigma u_\alpha \left\{1 - \dfrac{\Theta_m}{F(\alpha+m)} + \dfrac{\Theta_m^{\,2}}{F(\alpha+2m)\,F(\alpha+m)} - \&c.\right\}. \end{cases}$$

Let us now apply the method to the first example proposed, in its modified form, namely,

$$(xD)^2.y + xy = 0,$$

premising that its susceptibility of some such method of integration was suggested in the year 1847 by the Rev. Charles Graves, Professor of Mathematics in this University.

Operating on both sides of this equation with $\frac{1}{(xD)^2}$, we get

$$y + \frac{1}{(xD)^2}\, xy = C_1 \log x + C_2;$$

whence

$$y = \left\{1 - \frac{1}{(xD)^2}\, x + \frac{1}{(xD)^2}\, x \cdot \frac{1}{(xD)^2}\, x - \&c.\right\}.(C_1 \log x + C_2),$$

or

$$y = \begin{cases} C_1\left\{1 - x\,\dfrac{1}{(1+xD)^2} + x^2\,\dfrac{1}{(2+xD)^2\,(1+xD)^2} - \&c.\right\}\log x, \\ + \\ C_2\left\{1 - \dfrac{x}{1^2} + \dfrac{x^2}{1^2.2^2} - \dfrac{x^3}{1^2.2^2.3^2} + \dfrac{x^4}{1^2.2^2.3^2.4^2} - \&c.\right\}. \end{cases}$$

But

$$\frac{1}{(1+xD)^2}\log x = \frac{1}{1^2}\left(1 - \frac{2}{1}\, xD\right)\log x = \frac{1}{1^2}\,(\log x - 2),$$

$$\frac{1}{(2+xD)^2\,(1+xD)^2}\log x = \frac{1}{1^2.2^2}\left\{1 - 2\left(\frac{1}{1} + \frac{1}{2}\right)xD\right\}\log x$$

$$= \frac{1}{1^2.2^2}\left\{\log x - 2\left(\frac{1}{1} + \frac{1}{2}\right)\right\},\ \&c.$$

Hence, finally,

$$y = \begin{cases} (C_1 \log x + C_2)\left\{1 - \dfrac{x}{1^2} + \dfrac{x^2}{1^2.2^2} + \dfrac{x^3}{1^2.2^2.3^2} - \&c.\right\}, \\ + \\ 2C_1\left\{\dfrac{1}{1^2}\, x - \dfrac{1}{1^2.2^2}\left(\dfrac{1}{1} + \dfrac{1}{2}\right)x^2 + \dfrac{1}{1^2.2^2.3^2}\left(\dfrac{1}{1} + \dfrac{1}{2} + \dfrac{1}{3}\right)x^3 - \&c.\right\}. \end{cases}$$

If the equation to be integrated had been

$$xD^2.y + Dy + Mx^{m-1}y = 0,$$

its solution is obviously had by the same general method, and is

$$y = \begin{cases} C_1\left\{1 - Mx^m\,\dfrac{1}{(m+xD)^2} + (Mx^m)^2\,\dfrac{1}{(2m+xD)^2\,(m+xD)^2} - \&c.\right\}\log x, \\ + \\ C_2\left\{1 - \dfrac{Mx^m}{m^2} + \dfrac{(Mx^m)^2}{m^2.(2m)^2} - \dfrac{(Mx^m)^3}{m^2.(2m)^2.(3m)^2} + \&c.\right\}; \end{cases}$$

or

$$y = \begin{cases} (C_1 \log x + C_2)\left\{1 - \dfrac{Mx^m}{m^2} + \dfrac{(Mx^m)^2}{m^2.(2m)^2} - \dfrac{(Mx^m)^3}{m^2.(2m)^2.(3m)^2} + \&c.\right\} \\ + \\ 2C_1\left\{\dfrac{1}{m^2} Mx^m - \dfrac{m(2+1)}{m^3.(2m)^3}(Mx^m)^2 + \dfrac{m^2(6+3+2)}{m^3.(2m)^3.(3m)^3}(Mx^m)^3 - \&c.\right\} \end{cases}$$

Similarly, if it be proposed to integrate the partial differential equation

$$\nabla_2.z + \nabla.z + \Theta_m.z = 0,$$

where $$\nabla_2 = x^2 D_x^2 + 2xy D_x D_y + y^2 D_y^2,$$

a corresponding reduction gives

$$z + \frac{1}{\nabla^2}\Theta_m.z = u_0 \frac{\log x + \log y}{2} + v_0,$$

and the symbolic solution is

$$z = \left(1 - \frac{1}{\nabla^2}\Theta_m + \frac{1}{\nabla^2}\Theta_m.\frac{1}{\nabla^2}\Theta_m - \&c.\right)\left(u_0 \frac{\log x + \log y}{2} + v_0\right).$$

Hence

$$z = \begin{cases} u_0\left\{1 - \Theta_m \dfrac{1}{(m+\nabla)^2} + \Theta_m^2 \dfrac{1}{(m+\nabla)(2m+\nabla)^2} - \&c.\right\}\dfrac{\log x + \log y}{2}, \\ + \\ v_0\left\{1 - \dfrac{\Theta_m}{m^2} + \dfrac{\Theta_m^2}{m^2.(2m)^2} - \dfrac{\Theta_m^3}{m^2.(2m)^2.(3m)^2} + \&c.\right\}; \end{cases}$$

and finally,

$$z = \begin{cases} \left(u_0 \dfrac{\log x + \log y}{2} + v_0\right)\left\{1 - \dfrac{\Theta_m}{m^2} + \dfrac{\Theta_m^2}{1^2.2^2} - \dfrac{\Theta_m^3}{1^2.2^2.3^2} + \&c.\right\}, \\ + \\ 2u_0\left\{\dfrac{1}{m^2}\Theta_m - \dfrac{m(2+1)}{m^3.(2m)^3}\Theta_m^2 + \dfrac{m^2(6+3+2)}{m^3.(2m)^3.(3m)^3}\Theta_m^3 - \&c.\right\}. \end{cases}$$

Proceeding now to apply the same method to the modified form of the second example,

$$(xD-1)(xD+2)y + n^2x^2.y = 0,$$

we get

$$y + \frac{1}{(xD-1)(xD+2)} n^2x^2.y = C_1 x + \frac{C_2}{x^2}.$$

Consequently the solution is given by

$$y = \left\{1 - \frac{1}{(xD-1)(xD+2)}n^2x^2 + \frac{1}{(xD-1)(xD+2)}n^2x^2 \cdot \frac{1}{(xD-1)(xD+2)}n^2x^2 - \&c.\right\}\left(C_1x + \frac{C_2}{x^2}\right),$$

or

$$y = \begin{cases} C_1x\left\{1 - \dfrac{n^2x^2}{2.5} + \dfrac{(n^2x^2)^2}{2.5.4.7} - \dfrac{(n^2x^2)^3}{2.5.4.7.6.9} + \&c.\right\}, \\ + \\ \dfrac{a^2C^2}{1.2}\left\{1.2.\dfrac{1}{n^2x^2} + 1 - \dfrac{n^2x^2}{1.4} + \dfrac{(n^2x^2)^2}{1.4.3.6} - \&c.\right\}, \end{cases}$$

which may be condensed into the known form, but in its present shape can be generalized.

In connexion with the integration of equations with variable coefficients, and more particularly with the first example, the following important remark has been made by Gregory (*Examples*, p. 314). "After all, however, when these equations are of the second or higher orders, the number of cases in which they are integrable is very limited, and there seems to be no great prospect of the number being much increased. A little consideration will point out the reason of this. When we speak of an equation being integrable, we mean that the dependent variable can be expressed in terms of the independent variable by means of a finite series of functions of that quantity, the forms of such functions being limited to those known as algebraical and transcendental. Now it has been seen that the simplest forms of differential equations involve the highest transcendents which we recognize as known functions, such as e^{nx} or $\cos nx$; and it is to be expected, that when the equations become more complicated their integrals must involve higher transcendents to which we have not affixed particular names, and which we do not look on as known forms. This indeed is found to be the case, as for example in the equation

$$xD^2y + Dy + y = 0,$$

which in its integral involves the transcendent

$$\psi(x) = 1 - \frac{x}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2} + \&c.$$

It would appear then, that before we are able to make any further progress in the solution of differential equations,

we must create new transcendents in the same way as the ordinary transcendents e^x, $\cos x$, $\log x$, &c. have been created; we must study their properties, and endeavour to express the integrals of differential equations by means of them. The first part of this task has for some time past occupied the attention of mathematicians, and great progress has been made in it, though much still remains to be done. The second part has also been the object of study, though not to the same extent as the other; and several mathematicians have applied themselves with success to the expression of the integrals of differential equations by means of definite integrals which are the representatives of new transcendents. Thus, for instance, in the case cited above, the transcendent

$$1 - \frac{x}{1^2} + \frac{x^2}{1^2.2^2} - \frac{x^3}{1^2.2^2.3^2} + \&c. = \frac{1}{\pi}\int_0^{\pi} d\theta \cos(2 \sin\theta x^{\frac{1}{2}}).$$

Examples of such integrals will be found in Crelle's *Journal*, vol. x. p. 92; vol. xii. p. 144; vol. xvii. p. 363."

Now it appears to me that, until evaluated, the integral

$$\frac{C_2}{\pi}\int_0 d\theta \cos(2 \sin\theta x^{\frac{1}{2}})$$

must be considered to be quite as symbolic as the equivalent

$$\left\{1 + \frac{1}{(xD)^2}x\right\}^{-1}.C_2.$$

Indeed we may regard all symbolic condensations, as well as definite integrals, in the light of representatives of new transcendents.

For instance, if

$$U = A_0 + A_1 x + A_2 x^2 + \&c.,$$

where A_0, A_1, A_2, &c., are constants, it may easily be proved that

$$F(xD).U = F(0)\,A_0 + F(1)\,A_1 x + F(2)\,A_2 x^2 + \&c.$$

$$\frac{1}{F(xD)}.U = \frac{1}{F(0)}A_0 + \frac{1}{F(1)}A_1 x + \frac{1}{F(2)}A_2 x^2 + \&c.,$$

and it seems to me that the left-hand members may fairly be regarded as such representatives.

5, Trinity College, Dublin, March 1854.

ON THE CURVATURE OF CURVES IN SPACE.

By William Spottiswoode, M.A., F.R.S.

Let L, L_1, be the characteristics of the two surfaces, by the intersection of which the curve is formed; and let U, V, W, U_1, V_1, W_1, be the partial differential coefficients of L, L_1, with respect to x, y, z, respectively. Then writing

$$\left.\begin{aligned} &U : V : W \\ &= VW_1 - V_1W = WU_1 - W_1U : UV_1 - U_1V \end{aligned}\right\} \ldots (1),$$

the equations to a tangent line take the form

$$\left.\begin{aligned} &dx : dy : dz \\ &= U : V : W \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (2);$$

if θ be the common ratio of this system, there result

$$\left.\begin{aligned} \frac{d^2x}{ds^2} &= \frac{dU}{ds}\theta + U\frac{d\theta}{ds} \\ \frac{d^2y}{ds^2} &= \frac{dV}{ds}\theta + V\frac{d\theta}{ds} \\ \frac{d^2z}{ds^2} &= \frac{dW}{ds}\theta + W\frac{d\theta}{ds} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

whence

$$\begin{vmatrix} \frac{d^2x}{ds^2} & \frac{d^2y}{ds^2} & \frac{d^2z}{ds^2} \\ \frac{dU}{ds} & \frac{dV}{ds} & \frac{dW}{ds} \\ U & V & W \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots (4).$$

Now

$$\frac{dx}{ds}\frac{d^2x}{ds^2} + \frac{dy}{ds}\frac{d^2y}{ds^2} + \frac{dz}{ds}\frac{d^2z}{ds^2} = 0 \ldots\ldots\ldots (5),$$

or

$$U\frac{d^2x}{ds^2} + V\frac{d^2y}{ds^2} + W\frac{d^2z}{ds^2} = 0 \ldots\ldots\ldots\ldots (6);$$

whence, writing

$$U^2 + V^2 + W^2 = P^2 \ldots\ldots\ldots\ldots\ldots (7),$$

$$\frac{d^2x}{ds^2} : \frac{d^2y}{ds^2} : \frac{d^2z}{ds^2}$$

$$= P\frac{dU}{ds} - U\frac{dP}{ds} : P\frac{dV}{ds} - V\frac{dP}{ds} : P\frac{dW}{ds} - W\frac{dP}{ds} \quad \ldots(8);$$

or, writing

$$\left.\begin{aligned} \Omega &= \begin{vmatrix} U & V & W \\ U' & V' & W' \\ U_1 & V_1 & W_1 \end{vmatrix} = -(UU' + VV' + WW') \\ \Omega_1 &= \begin{vmatrix} U_1 & V_1 & W_1 \\ U_1' & V_1' & W_1' \\ U & V & W \end{vmatrix} = UU_1' + VV_1' + WW_1' \end{aligned}\right\} \ldots(9),$$

these ratios

$$\left.\begin{aligned} &= (W\Omega_1 + W_1\Omega)V - (V\Omega_1 + V_1\Omega)W \\ &: (U\Omega_1 + U_1\Omega)W - (W\Omega_1 + W_1\Omega)U \\ &: (V\Omega_1 + V_1\Omega)U - (U\Omega_1 + U_1\Omega)V \end{aligned}\right\} \ldots\ldots(10);$$

or, if ϕ be the angle between the two surfaces,

$$\left.\begin{aligned} &= \Omega_1 P(PU_1 - P_1U\cos\phi) + \Omega P_1(P_1U - PU_1)\cos\phi \\ &: \Omega_1 P(PV_1 - P_1V\cos\phi) + \Omega P_1(P_1V - PV_1)\cos\phi \\ &: \Omega_1 P(PW_1 - P_1W\cos\phi) + \Omega P_1(P_1W - PW_1\cos\phi) \end{aligned}\right\} \ldots(11).$$

But if ρ, ρ_1, be the radii of curvature of normal sections of the two surfaces passing through the tangent line to the curve, it will be found that

$$\Omega : \Omega_1 = \frac{P}{\rho} : \frac{P_1}{\rho_1} \ldots\ldots\ldots\ldots\ldots(12),$$

so that (11) may be again transformed into the following:

$$\left.\begin{aligned} &\left(\frac{1}{\rho_1\cos\phi} - \frac{1}{\rho}\right)PU_1 + \left(\frac{1}{\rho\cos\phi} - \frac{1}{\rho_1}\right)P_1V \\ &: \left(\frac{1}{\rho_1\cos\phi} - \frac{1}{\rho}\right)PV_1 + \left(\frac{1}{\rho\cos\phi} - \frac{1}{\rho_1}\right)P_1V \\ &: \left(\frac{1}{\rho_1\cos\phi} - \frac{1}{\rho}\right)PW_1 + \left(\frac{1}{\rho\cos\phi} - \frac{1}{\rho_1}\right)P_1W \end{aligned}\right\} \ldots(13).$$

Now $\rho\cos\phi$ is the radius of curvature of the oblique section of the first surface, made by the normal plane to

the second which passes through the tangent line; so that $\frac{1}{\rho\cos\phi} - \frac{1}{\rho_1}$ is the difference of the curvatures of the two curves formed by the intersection of that plane with the two surfaces; similarly $\frac{1}{\rho_1\cos\phi} - \frac{1}{\rho}$ is the difference of the curvatures of the two curves formed by the intersection of the normal plane to the first surface, which passes through the tangent line to the given curve, with the two surfaces themselves.

If ρ be the radius of absolute curvature of the given curve,

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2 \ldots\ldots\ldots\ldots(14),$$

and consequently

$$C = \frac{1}{\rho\cos\phi} - \frac{1}{\rho_1}, \quad C_1 = \frac{1}{\rho_1\cos\phi} - \frac{1}{\rho} \ldots\ldots(15),$$

the common ratio of the systems (8) and (13) becomes

$$= \rho\mathrm{P}\mathrm{P}_1(C^2 + 2CC_1\cos\phi + C_1^2)^{\frac{1}{2}} \ldots\ldots\ldots\ldots(16);$$

or again,

$$= \rho^2 PP_1\left(\frac{C}{\rho} + \frac{C_1}{\rho_1}\right):$$

hence

$$\frac{1}{\rho} = \frac{1}{\sin\phi}\left\{\frac{1}{\rho^2} + \frac{1}{\rho_1^2} - \frac{2\cos\phi}{\rho\rho_1}\right\}^{\frac{1}{2}} \ldots\ldots\ldots\ldots(17),$$

which is an expression for the radius of absolute curvature of the curve in terms of the radii of curvature of the two surfaces. When the surfaces touch one another

$$\sin\phi = 0, \quad \cos\phi = 1,$$

and consequently $\rho = 0$, unless $\rho = \rho_1$; that is to say (as will be found on substituting the ordinary values of ρ and ρ_1), unless for some normal section $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, are the same for both surfaces; in other words, unless the surfaces touch in more points than one. This might have been anticipated *à priori*, since the curve would otherwise reduce itself to a single point. If $\rho = \rho_1$ the value of ρ becomes indeterminate, because the condition $\rho = \rho_1$ indicates that the radii of curvature of *any* common normal section of the two surfaces passing through the point in question are equal, and as this curvature may vary with the direction

of the normal plane, there are in fact an infinite number of branches of the curve passing through the point, each of which may have its own curvature. And on referring to (17) it will be found that

$$\frac{2}{\rho} = \frac{1}{\rho} + \frac{1}{\rho_1},$$

and consequently if for any one or more sections

$$\rho = \rho_1,$$

(*i.e.* if along any one or more sections the surfaces have their convexities turned in the same direction,)

$$\rho = \rho_1 = \rho = 1,$$

i.e. the radius of absolute curvature of that branch of the curve is the same as the common radius of curvature of the normal section of the two surfaces on which that branch lies; but if for any one or more sections,

$$\rho + \rho_1 = 0,$$

i.e. if along any one or more sections the surfaces have their convexities turned in different directions, *i.e.*

$$\rho = 0,$$

either there is no branch in that direction, or the branch has a point of inflexion or of suspended curvature.

If the two surfaces cut one another at right angles

$$\sin\phi = 1, \quad \cos\phi = 0,$$

and consequently

$$\frac{1}{\rho^2} = \frac{1}{\rho^2} + \frac{1}{\rho_1^2}.$$

It is well known that if three surfaces cut one another at right angles they cut one another in their principal sections and in their lines of curvature; hence, if ρ, ρ_1, ρ_2, be the radii of absolute curvature of the three curves so formed, and R, R^1, R_1, R_1^1, R_2, R_2^1, the principal radii of curvature of the three surfaces respectively, there will result

$$\frac{1}{\rho^2} = \frac{1}{R_1^2} + \frac{1}{R_2'^2}, \quad \frac{1}{\rho_1^2} = \frac{1}{R_2^2} + \frac{1}{R'^2}, \quad \frac{1}{\rho_2^2} = \frac{1}{R^2} + \frac{1}{R_1'^2},$$

and consequently

$$\frac{1}{\rho^2} + \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} = \frac{1}{R^2} + \frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R'^2} + \frac{1}{R_1'^2} + \frac{1}{R_2'^2}.$$

In the case of an umbilicus

$$R = R^1, \quad R_1 = R_1^1, \quad R_2 = R_2^1;$$

and consequently, writing

$$\frac{1}{\rho} + \frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{2}{\Sigma},$$

$$\frac{1}{R^2} = 2\left(\frac{1}{\rho_1} - \frac{1}{\Sigma}\right)\left(\frac{1}{\rho_2} - \frac{1}{\Sigma}\right),$$

$$\frac{1}{R_1^2} = 2\left(\frac{1}{\rho_2} - \frac{1}{\Sigma}\right)\left(\frac{1}{\rho} - \frac{1}{\Sigma}\right),$$

$$\frac{1}{R_2^2} = 2\left(\frac{1}{\rho} - \frac{1}{\Sigma}\right)\left(\frac{1}{\rho_1} - \frac{1}{\Sigma}\right);$$

which may be written also as follows,

$$\frac{1}{R^2}\left(\frac{1}{\rho} - \frac{1}{\Sigma}\right) = \frac{1}{R_1^2}\left(\frac{1}{\rho_1} - \frac{1}{\Sigma}\right) = \frac{1}{R_2^2}\left(\frac{1}{\rho_2} - \frac{1}{\Sigma}\right)$$

$$= -\frac{1}{(R^2 + R_1^2 + R_2^2)\Sigma}.$$

ON THE CONJUGATE LINES OF SURFACES.

By W. Walton.

Let $F = 0$ be the algebraical equation to a surface, free from radical and negative indices. In elementary works on Solid Geometry (see Gregory's *Solid Geometry*, chap. XIII.) it is shewn that the existence of a tangent cone at any point of a surface, or of an edge of regression, or in fact, generally, of plural tangency, corresponds to the coexistence of the four equations

$$F = 0, \quad \frac{dF}{dx} = 0, \quad \frac{dF}{dy} = 0, \quad \frac{dF}{dz} = 0.$$

It appears however to have escaped the notice of writers on Geometry, that these analytical conditions sometimes correspond to the existence of a *conjugate curved line*, that is, to a curved line entirely detached from the surface, but forming part of the geometrical locus of its equation, bearing in fact the same relation to the surface which a conjugate point bears to a curve.

Imagine the existence of a conjugate line: let x, y, z, be the coordinates of any one of its points. Suppose x_1, y_1, z_1, to be written in the equation $F = 0$ in lieu of x, y, z, y_1 and z_1 being coordinates of a point in the plane of y, z, nearly but not quite in contact with the projection of the conjugate curve upon that plane. Then x_1 in the equation $F(x_1, y_1, z_1) = 0$ will be impossible, and therefore have two values of the forms

$$\alpha + \beta\sqrt{(-1)}, \quad \alpha - \beta\sqrt{(-1)}.$$

Hence, when one of these roots of the equation in x_1 becomes α, the other will also become α. Thus we see that in the equation

$$F(x, y, z) = 0$$

x will have two equal values for every pair of values of y, z. Hence

$$\frac{dF}{dx} = 0.$$

Similarly we see that

$$\frac{dF}{dy} = 0, \quad \frac{dF}{dz} = 0.$$

Thus the existence of a conjugate line implies the coexistence of the four equations

$$F = 0, \quad U = 0, \quad V = 0, \quad W = 0,$$

where U, V, W, denote the partial differential coefficients of F with regard to x, y, z, respectively.

Putting

$$\frac{d^2F}{dx^2} = u, \quad \frac{d^2F}{dy^2} = v, \quad \frac{d^2F}{dz^2} = w,$$

$$\frac{d^2F}{dy\,dz} = u', \quad \frac{d^2F}{dz\,dx} = v', \quad \frac{d^2F}{dx\,dy} = w',$$

we have, twice differentiating the equation to the surface, and writing λ, μ, ν, for the direction-cosines of the tangent line at any point of the conjugate line,

$$\lambda^2 u + \mu^2 v + \nu^2 w + 2\mu\nu u' + 2\nu\lambda v' + 2\lambda\mu w' = 0 \ldots (A).$$

Since each of the ratios between the quantities λ, μ, ν, can have only one value for a given point x, y, z, in the conjugate line, the algebraical state of this equation must be such as to establish this restriction. Thus we see that

the sufficient and necessary conditions for the existence of a conjugate line are the coexistence of the four equations

$$F = 0, \quad U = 0, \quad V = 0, \quad W = 0,$$

and the singleness of values of the ratios between λ, μ, ν, as implied by the equation (A).

Ex. Take the equation

$$F = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)^2 - \left(\frac{xyz}{m^3} - 1\right)^2 \cdot \left(\frac{xyz}{n^3} - 1\right),$$

where n is supposed to be greater than m.

It is easily ascertained that the conditions

$$F = 0, \quad U = 0, \quad V = 0, \quad W = 0,$$

are satisfied by the two simultaneous equations

$$xyz = m^3,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and that, under these circumstances, $\frac{1}{k^6}$ being substituted for $\frac{n^3 - m^3}{m^6 n^3}$,

$$u = \frac{8x^2}{a^4} + \frac{2y^2z^2}{k^6},$$

$$v = \frac{8y^2}{b^4} + \frac{2z^2x^2}{k^6},$$

$$w = \frac{8z^2}{c^4} + \frac{2x^2y^2}{k^6},$$

$$u' = \frac{8yz}{b^2c^2} + \frac{2x^2yz}{k^6}$$

$$v' = \frac{8zx}{c^2a^2} + \frac{2y^2zx}{k^6},$$

$$w' = \frac{8xy}{a^2b^2} + \frac{2z^2xy}{k^6}.$$

Hence the equation (A) becomes

$$4\left(\frac{\lambda x}{a^2} + \frac{\mu y}{b^2} + \frac{\nu z}{c^2}\right)^2 + \frac{1}{k^6}(\lambda yz + \mu zx + \nu xy)^2 = 0,$$

which can be satisfied only by putting

$$\frac{\lambda x}{a^2} + \frac{\mu y}{b^2} + \frac{\nu z}{c^2} = 0,$$

and

$$\frac{\lambda}{x} + \frac{\mu}{y} + \frac{\nu}{z} = 0.$$

Thus the ratios between λ, μ, ν, have only single values, and therefore the curve of intersection of the two surfaces, represented by the equations

$$xyz = m^3, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

is a conjugate or isolated line.

12*th July*, 1854.

THE GEOMETRY OF QUATERNIONS.

By JOHN PATERSON, of Albany (N. Y.)

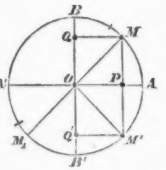
On the Extension of the principle of Perpendicularity to the Three dimensions of Space.

1. THE remarkable discovery of a new analytical element, denominated *quaternion*, by Sir W. R. Hamilton of Dublin, appears destined to mark an epoch in the history of geometry. As the discovery consists essentially in an extension of the principle of perpendicularity from a single plane, to the system of three mutually rectangular planes that serve for the admeasurement of the three dimensions of space, it may be of use to investigate the genesis of the diagonal of a rectangular parallelopipedon, by the same kind of operations that generate the diagonal of a rectangular parallelogram.

2. The principle of perpendicularity consists in the dynamico-geometrical fact, that a straight line, considered as the measure of the result of an operation, has at the same time two distinct values, namely, the value of its own magnitude or length in the direction in which it lies, and the value zero in the direction perpendicular thereto. As an immediate consequence of this fact, if the line be inclined to the system of two rectangular axes to which it is referred for measurement, it will have a finite value on each axis, equal to its projection thereon; that is, its product by the

cosine of the angle it makes with that axis, which again is equal to its product by the sine it makes with the other axis.

If we regard such line, in the different positions which may be given to it about an origin in a plane, when referred to a fixed system of rectangular axes, as the measure of an operation of the first order, such as that of simple addition or of simple multiplication, by a linear factor, it can be readily understood either as the measure of a rectilinear force of the first order acting in its direction, and beginning at the origin O; or as the measure of the result of a circular force of the same order, producing uniform rotation about the centre O, and beginning at the axis OA. In both cases, the line in the position OM is measured at once by its projections OP and OQ: in the former case, OM is the measure both of the operation (which consists in performing a multiplication by this line OM; 1_μ being multiplicand and $1_\lambda = OM$ multiplier, $1_\lambda.1_\mu = OM \times 1_\mu$), and of the result of the operation; but in the latter case, the operation consists in performing a multiplication by the angle AOM or its measure the arc $AM = \theta.1_r$, the radius $OA = 1_r$ (or even $1_r.1_\mu$) being the multiplicand, $\theta.1_r.1_\epsilon.1_\mu = AM \times OA \times 1_\mu = OM \times 1_\mu$; so that here the measure of the result OM differs from the measure of the operation itself, which is the arc AM; but the measure of the result, $QM = OP + OQ$, is all the notation records.



3. If we begin with a linear multiplication at the origin O, the indeterminate radius OA may be adopted as linear unit $1_\epsilon = 1_\lambda$ when written as a multiplier; and 1_μ being multiplicand, $1_\lambda.1_\mu$ expresses the transfer of 1_μ from OA, in which position it is usually written $+1.1_\epsilon.1_\mu = OA \times 1_\mu$, where $+1$ is the coefficient of a positive linear operation, and $+1.1_\epsilon$ is the measure of such operation; which notation stands directly contrasted with $-1.1_\lambda.1_\mu = -1.1_\epsilon.1_\mu = OA' \times 1_\mu$, as representing the transfer of 1_μ from O to A', a negative linear operation, having -1.1_ϵ as its measure, and -1 as coefficient of the same.

Next taking $OA \times 1_\mu = +1.1_\epsilon.1_\mu$ as multiplicand, and the arc $AM = \theta.1_r$ as multiplier, the product $\theta.1_r.1_\epsilon.1_\mu = OM \times 1_\mu$; in which position $OM = AM \times 1_\epsilon = \theta.1_r.1_\epsilon$ has the two mea-

sures $OP = +1.\cos\theta.1_\epsilon$ and $OQ = +\sqrt{(-1)}\sin\theta.1_\epsilon$, the last notation for a reason that will presently appear. A second multiplication by any new arc whatever, $\theta'.1_\tau = MN$, will transfer the radius into the position ON, where its measures on the horizontal and perpendicular axes are given by the expressions $\cos(\theta+\theta')\,1_\epsilon$ and $\sin(\theta+\theta')\,1_\epsilon$ (the last affected by the symbol $\sqrt{(-1)}$, as may be traced upon the diagram.

4. If $\theta' = \theta = \frac{1}{2}\pi = 90°$, a right angle, the first multiplication will transfer the loaded radius or multiplicand $1_\epsilon.1_\mu = 1_\tau.1_\mu$ from the position OA to the position OB; whence the second multiplication carries it to the position OA', where it is expressed by the notation $-1.1_\epsilon.1_\mu$, the measure of the *result* of a negative operation; and since $\frac{1}{2}\pi.1_\tau = 90°.1_\tau$ is the factor that gives the position OB by a simple multiplication, and the result $OA' = -1.1_\epsilon$ when the multiplication is repeated (the square of the operation), it is plain that the former result may be denoted as the square root of the latter, that is, $OB = \sqrt{(-1)}\,1_\epsilon$. On this convention, however, it is to be borne in mind that when the symbol $\sqrt{(-1)}$ is the coefficient of a multiplier, it governs a right angle, or the arc of 90°; but when it expresses a product or result, it is the unit coefficient of a straight line perpendicular to the rectilinear multiplicand to which it has been applied to produce that result. Thus, when applied to the multiplicand 1_ϵ in the position $OA = +1.1_\epsilon$, the multiplier $\sqrt{(-1)}\,1_\tau = \text{arc}\,AB = 90°$ (= the measure of the operation) gives the product $OB = +\sqrt{(-1)}\,1_\epsilon$ (= the measure of the result of the operation): a second application of the multiplier $\sqrt{(-1)}\,1_\tau$ gives the result $OA' = -1.1_\epsilon$; a third application, the result $OB' = -\sqrt{(-1)}\,1_\epsilon$; and a fourth application, the result $OA = +1.1_\epsilon$, identical with the primitive position.

5. θ being any arc whatever AM, if θ' be its conjugate or complementary arc to the circumference, the product $\theta'.\theta.1_\tau.1_\epsilon.1_\mu$ transfers $1_\epsilon.1_\mu$ from OA to OM by the first multiplier $\theta.1_\tau$, and from OM through $MA'A$ to OA by second multiplier $\theta'.1_\tau$. Since this gives $\theta'.\theta.1_\tau.1_\epsilon.1_\mu = +1.1_\epsilon.1_\mu$, we have $\theta'.\theta.1_\tau = +1$, and therefore $\theta' = \dfrac{1}{\theta}$. The conjugate factor $\dfrac{1}{\theta}.1_\tau$ then represents the multiplying arc $MBA'B'A$, complementary of AM to the circumference, and passes the radius through the former to its primitive position OA; and

as the same result would be given by retrograde rotation from M to A, it is seen that the negative factor $-\theta.1_r$ produces the same result as the positive factor $\frac{1}{\theta}.1_r$, but through a different path of operation.

6. The negative arc $-\theta.1_r$ will be represented by AM'; and as $OQ = +\sqrt{(-1)}\sin\theta.1_e$ is the sine of the positive arc AM, we have $OQ' = -\sqrt{(-1)}\sin\theta.1_e$ for the sine of the negative arc AM'. Taking the general unit radius 1_r (irrespective of position), we can write

$$\{1+\sqrt{(-1)}\sin\theta\}\,1_r = B'O + OQ = B'Q,$$

and

$$\{1-\sqrt{(-1)}\sin\theta\}\,1_r = B'O - Q'O = B'Q'.$$

The product of these factors is thus effected, term by term: $\{1+\sqrt{(-1)}\sin\theta\}\,1_e$ being the multiplicand or passive factor, the term 1_e is carried by the first term $+1.1_r$ of the multiplier or active factor, through the circumference, which is a perfect natural unit of space, back to its primitive position $B'O$, thereby fulfilling the condition $1^2.1_e = 1_e$; and similarly the same unit multiplier carries the term $+\sqrt{(-1)}\sin\theta.1_e$ through the circumference, back to its primitive position OQ. The second term $-\sqrt{(-1)}\sin\theta.1_r$ of the multiplier carries the radius 1_e from the position OA (for 1.1_e is completely indeterminate in its first position), through three right angles, into the position OB', and reduces its magnitude to $\sin\theta.1_e$; and finally, the same multiplying term, applied to the passive term $+\sqrt{(-1)}\sin\theta.1_e = OQ$, carries it, through three right angles, into the position OA, and reduces its magnitude to $\sin^2\theta.1_e$. The complete product then stands

$$\{1^2+\sqrt{(-1)}\sin\theta-\sqrt{(-1)}\sin\theta+\sin^2\theta\}\,1_e = (1+\sin^2\theta)\,1_e,$$

which may be immediately constructed for any unit radius and any arc θ; or if θ be determined by the condition that $1+\sin^2\theta = 0$, we find that $\sin\theta = \pm\sqrt{(-1)}$, which requires that $\theta = \pm\frac{1}{2}\pi$. Instead of the unit radius $1_r = 1_e$, $\cos\theta$ might be taken for the first term of both factors, and then the multiplication gives

$$\{\cos\theta+\sqrt{(-1)}\sin\theta\}\,\{\cos\theta-\sqrt{(-1)}\sin\theta\} = \cos^2\theta+\sin^2\theta,$$

a result otherwise known to be equal to the square of unity; and in this way we obtain a demonstration of the forty-seventh proposition of the first book of Euclid, by the method of the calculus of operations. Our demonstration, however, is more general than that of Euclid; for in the

complete equation $(a^2+b^2)1_e^2 = R^2 1_e^2$ (where 1_e is the linear unit, and $R1_e$, $a1_e$, and $b1_e$ the hypothenuse and sides of a right-angled triangle), the *spacial* factor $1_e \times 1_e$ may have many different forms, while in Euclid it represents an area only.

7. As $1^2 = 1$, if the multiplicand be the linear unit 1_e, and the multiplier the abstract number 1, we have

$$(\cos^2\theta + \sin^2\theta)\,1_e = 1^2.1_e = 1_e = OA = OM$$

the diagonal of the rectangular parallelogram, of which the sides are $OP = \cos\theta.1_e$ and $OQ = \sin\theta.1_e$, where the coefficients $\cos\theta$ and $\sin\theta$ are numerical, and the products $\cos\theta.1_e$ and $\sin\theta.1_e$ are *magnitudinal* only; but as also $(+1)^2 = +1$, we have

$$\{\cos\theta + \sqrt{(-1)}\sin\theta\}\,\{\cos\theta - \sqrt{(-1)}\sin\theta\}\,1_e = (+1)^2\,1_e = +1.1_e$$
$$= OA = OM \times OM',$$

the product of the two diagonals whose sides are respectively

$$OP = +1\cos\theta.1_e \text{ and } OQ = +\sqrt{(-1)}\sin\theta.1_e,$$

and $$OP = +1\cos\theta.1_e \text{ and } OQ' = -\sqrt{(-1)}\sin\theta.1_e,$$

where the *directional* coefficients $+1$ and $\sqrt{(-1)}$ also appear, and the complete terms involve both magnitude and direction.

8. The diagonal of a rectangular parallelopipedon furnishes the spacial analogue of the preceding formula, and is expressed by

$$(\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')\,1_e = (+1)^2\,1_e = +1.1_e;$$

θ', θ'', θ''', being arcs of mutually perpendicular great circles, of the sphere of which the unit radius $1_r = 1_e$ forms the diagonal of the parallelopipedon.

If R be any arithmetical coefficient, and $R1_e$ the magnitude of the diagonal of a rectangular parallelopipedon, the equation

$$R^2(\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')\,1_e = +1.R^2 1_e$$

denotes a concrete biquaternion in its most general form, and is the product of the two simple concrete quaternions denoted by the equations

$$R\{+\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}\,1_e = +1.R1_e, \text{ and}$$

$$R\{-\sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta'' - \sqrt{(-1)}\sin\theta'''\}\,1_e = \frac{1}{+1}.R1_e = +1.R1_e,$$

the latter being the conjugate or reciprocal of the former, and each the diagonal of a rectangular parallelopipedon. But as the directional coefficient is wholly determined within the sphere to radius unity, we may make $R = 1$, and generate the biquaternion unity by the actual multiplication of the left-hand members of these two equations. That this operation really consists in extending the application of the principle of perpendicularity from a single plane to the three mutually perpendicular coordinate planes of general space, must appear through the following investigations.

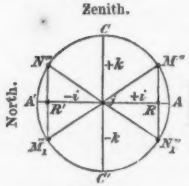
9. Let $OM = 1_r = 1_e$ be the unit radius of a sphere, situated in the superior south-west quadrant formed by the mutual intersections of the horizontal, meridional, and equatorial planes; and let its projections on the three positive axes (western, southern, and ascending) be OP, OQ, and OR. Let three planes be drawn through OM, perpendicular respectively to the three coordinate axes; and draw radii to the respective intersections M', M'', and M''', of these planes with the sides of one triquadrantal triangle ABC formed by the circumferences of the horizontal, meridional, and equatorial circles: each angle of each of these three radii with the axes will be equal to the angle of OM with the same axis, and hence arise the following fundamental relations of admeasurement:

$$OP = OM' \times \sin AOM' . 1_e = + \sqrt{(-1)} \sin \theta' . 1_e,$$
$$OQ = OM'' \times \sin BOM'' . 1_e = + \sqrt{(-1)} \sin \theta'' . 1_e,$$
$$OR = OM''' \times \sin COM''' . 1_e = + \sqrt{(-1)} \sin \theta''' . 1_e;$$

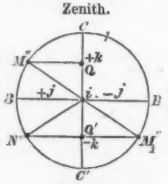
which immediately point out the following as their conjugates:

$$OP' = ON' \times \sin AON' . 1_e = - \sqrt{(-1)} \sin \theta' . 1_e,$$
$$OQ' = ON'' \times \sin BON'' . 1_e = - \sqrt{(-1)} \sin \theta'' . 1_e,$$
$$OR' = ON''' \times \sin CON''' . 1_e = - \sqrt{(-1)} \sin \theta''' . 1_e.$$

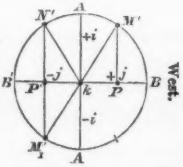
III. *Meridional Plane.* FACE WEST. Zenith. North. South. Nadir.



II. *Equatorial Plane.* FACE SOUTH. Zenith. West. East. Nadir.



I. *Horizontal Plane.* FACE ZENITH. South. West. North.



First diagram. A, the southernmost point of the horizontal plane, is the origin of arcs θ', and the rotation is right-handed through B, &c.

Second diagram. B, the westernmost point of the equatorial plane, is the origin of arcs θ'', and the rotation is right-handed through C, &c.

Third diagram. C, the highest point of the meridional plane, is the origin of arcs θ''', and the rotation is right-handed through A, &c.

These several origins and directions of operation establish, by aid of the principle of perpendicularity, the following measures of the results of quadrantal steps of rotation:

I.	II.	III.
$OA = +1'.1_\varepsilon = +i;$	$OB = +1''.1_\varepsilon = +j;$	$OC = +1'''.1_\varepsilon = +k;$
$OB = +\sqrt{(-1)'}.1_\varepsilon = +j;$	$OC = +\sqrt{(-1)''}.1_\varepsilon = +k;$	$OA = +\sqrt{(-1)'''}.1_\varepsilon = +i;$
$OA' = -1'.1_\varepsilon = -i;$	$OB' = -1''.1_\varepsilon = -j;$	$OC' = -1'''.1_\varepsilon = -k;$
$OB' = -\sqrt{(-1)'}.1_\varepsilon = -j.$	$OC' = -\sqrt{(-1)''}.1_\varepsilon = -k.$	$OA' = \sqrt{(-1)'''}.1_\varepsilon = -i.$

10. The multiplication of the two simple quaternions

$$\{+\sqrt{(-1)}\sin\theta'+\sqrt{(-1)}\sin\theta''+\sqrt{(-1)}\sin\theta'''\}1_\varepsilon = +1.1_\varepsilon = OM, \text{ and}$$

$$\{-\sqrt{(-1)}\sin\theta'-\sqrt{(-1)}\sin\theta''-\sqrt{(-1)}\sin\theta'''\}1_\varepsilon = \frac{1}{+1}.1_\varepsilon = ON \text{ or } OM,$$

(for the right-hand member is really indeterminate in its direction, and may have any required position to satisfy the conditions of the operation), is now to be effected term by term, to shew the genesis of the biquaternion

$$(\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')\,1_\varepsilon = (+1)^2.1_\varepsilon = +1.1_\varepsilon = OM.$$

The notation of the process with the left-hand members stands thus:

$$\begin{array}{l} +\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta''' \\ -\sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta'' - \sqrt{(-1)}\sin\theta''' \\ \hline +1\sin^2\theta' \quad +1\sin^2\theta'' \quad +1\sin^2\theta''' \end{array}$$

$$\left\{\begin{matrix}\{+\sqrt{(-1)}\sin\theta'\}\{-\sqrt{(-1)}\sin\theta''\}\\ \{+\sqrt{(-1)}\sin\theta''\}\{-\sqrt{(-1)}\sin\theta'\}\end{matrix}\right\} + \left\{\begin{matrix}\{+\sqrt{(-1)}\sin\theta'''\}\{-\sqrt{(-1)}\sin\theta'\}\\ \{+\sqrt{(-1)}\sin\theta'\}\{-\sqrt{(-1)}\sin\theta'''\}\end{matrix}\right\}$$

$$+ \left\{\begin{matrix}\{+\sqrt{(-1)}\sin\theta''\}\{-\sqrt{(-1)}\sin\theta'''\}\\ \{+\sqrt{(-1)}\sin\theta'''\}\{-\sqrt{(-1)}\sin\theta''\}\end{matrix}\right\}.$$

The operations may be as follow:

(1) The multiplicand 1_e has the value $+1'.1_e$ in the position OA; and the application of the multiplier $+\sqrt{(-)}\sin\theta'.1_r$ carries it right-handedly, through a right angle, into the position OB, at the same time reducing its value to

$$+\sqrt{(-1)}\sin\theta'.1_e = OP.$$

The second multiplier $-\sqrt{(-1)}\sin\theta'.1_r$ being now applied, carries the new multiplicand $+\sqrt{(-1)}\sin\theta'.1_e$ through three right angles, and reduces its value to $+1.\sin^2\theta'.1_e$ upon the axis OA. As the spacial multiplier in each instance is really the angle or arc, the *directional* effects alone of these two multiplications are equivalent to the rotation of the radius $1_e = OA$, first through the angle $\theta' = AOM'$ (the measure of which is the arc $\theta'.1_r = AM'$), and then through the conjugate angle $M'OBA'B'A$, the measure of which is the reciprocal arc

$$\frac{1}{\theta'}.1_r = (2\pi - \theta')1_r = M'BA'B'A = ABA'B'N',$$

of which the sine is $-\sqrt{(-1)}\sin\theta'.1_r = OP'$.

(2) The multiplicand 1_e has the value $+1''.1_e$ in the position OB; and the multiplier $+\sqrt{(-1)}\sin\theta''.1_r$ (measure of the arc $\theta''.1_r = BM''$) will carry it right-handedly through a right angle, reducing its value to $+\sqrt{(-1)}\sin\theta''.1_e = OQ$ upon the radius OC. From this position, the second multiplier $-\sqrt{(-1)}\sin\theta''.1_r$ carries the new multiplicand $+\sqrt{(-1)}\sin\theta''.1_e$ through three right angles, and reduces its value to $+1.\sin^2\theta''.1_e$ upon the axis OB. In directional effect alone, the two multiplications are equivalent to the multiplication of the radius $1_e = OB$, first by the arc $\theta''.1_r = BM''$, and then by its reciprocal $\frac{1}{\theta''}.1_r = (2\pi-\theta'')1_r = M''CB'C'B = BCB'C'N''$, the sine of which is $OQ' = -\sqrt{(-1)}\sin\theta''.1_r$.

(3) Lastly, the multiplicand 1_e has the value $+1'''.1_e$ in the position OC; whence the multiplier $+\sqrt{(-1)}\sin\theta'''.1_r$ (measure of the arc $\theta'''.1_r = CM'''$) transfers it through a right angle, and reduces its value to $+\sqrt{(-1)}\sin\theta'''.1_e = OR$ upon the radius OA. From this position, the second multiplier $-\sqrt{(-1)}\sin\theta'''.1_r$ carries it through three right angles, and reduces its value to $+1.\sin^2\theta'''.1_e$ upon the axis OC. On directional operation, the two are equivalent to the multiplication of the radius $1_e = OC$, first by the arc $\theta'''.1_r = CM'''$,

and then by its reciprocal $\frac{1}{\theta'''}.1_r = M'''AC'A'C = CAC'A'N'''$,

the sine of which is $OR' = -\sqrt{(-1)}\sin\theta'''.1_r$.

The sum of the results of these three operations of multiplication is therefore $(+\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')1_e$, which is otherwise known to be equal to 1.1_e; and this disposes of the first line of one product, leaving the second line (a double one, enclosed in braces) to be examined.

(1) The multiplicand 1_e is carried by the factor $+\sqrt{(-1)}\sin\theta'.1_r$, from OA (fig. I.), through a right angle, into the position OB; whence the factor $-\sqrt{(-1)}\sin\theta''.1_r$ carries it through three right angles, upon OC' (fig. II.). The multiplying angles being here mutually perpendicular, their product does not give the result $+1 = \{+\sqrt{(-1)}\}\{-\sqrt{(-1)}\}$; but the result $\{+\sqrt{(-1)}\}\{-\sqrt{(-1)}\}\sin\theta'.\sin\theta''.1_e$ is a measure of the radius $OM_1 = -1.1_e$ on the negative axis OC'; and similarly it is found that the double factor in the first line comprised in each of the two other pairs of braces, gives a measure of OM_1 on a negative axis, namely, one on OB' and one on OA'; so that these three partial measures form the complete measure of $OM_1 = -1.1_e$, as the result of the factors written in the first line between the braces.

(2) Then in the second line, the multiplicand 1_e is carried by the factor $+\sqrt{(-1)}\sin\theta''.1_r$, from OC' (fig. II.), through one right angle, into the position OB (fig. I.); whence the factor $-\sqrt{(-1)}\sin\theta'.1_r$ carries it through three right angles, upon OA, where it constitutes a first partial measure of $OM = +1.1_e$. The double factor of the second line in each of the two other braces similarly gives a partial measure of OM on a positive axis, namely, one on OC and one on OB; so that all three together form the complete measure of $OM = +1.1_e$, and the results of the two lines between the braces balance in the equation $-1.1_e + 1.1_e = 0$.

11. The right-hand member of the equation of quaternions denotes the radius independent of direction, while the several terms of the left-hand member denote the respective measures of the radius upon that axis which is perpendicular to the axis from which its angle is counted. The diagonal OM of the parallelopipedon has the three measures OP, OQ, and OR, which are severally the sines of the arcs AM', BM'', CM''', that serve to express the primitive quaternion

$$\{+\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}\,1_e = 1_e = OM.$$

The radii OM', OM'', OM''', are obtained by revolving OM perpendicularly to each coordinate plane respectively, until its intersection with the same.

If a great circle be drawn through the points A and M, and $\alpha.1_r$ be written for the arc AM that measures the angle AOM, the reciprocal of AM will be the arc $\frac{1}{\alpha}.1_r = AMA'N$ that measures the angle $(2\pi - \alpha) = AOMA'N = MOA'NA$. The radius ON would represent the quaternion conjugate to OM; but this radius ON, being revolved perpendicularly towards each coordinate plane till it meets the same, will determine three radii ON', ON'', and ON''' (the last being substituted for ON_1, because θ''' *decreases* as α *increases*), whose measures are seen to be respectively OP', OQ', and OR', which are precisely the measures of the radii OM_1', OM_1'', and OM_1''', respectively the opposites of the radii OM', OM'', and OM''' that constitute the primitive quaternion OM; and finally these radii OM_1', OM_1'', and OM_1''', are severally determined from OM_1, the opposing radius to OM, just as OM', OM'', and OM''' were determined from OM, and ON', ON'', and ON''' from ON, being the projections of OM_1 and ON upon the negative axes. Then since it has been seen that $OP' = -\sqrt{(-1)}\sin\theta'.1_e$, $OQ' = -\sqrt{(-1)}\sin\theta''.1_e$, and $OR' = -\sqrt{(-1)}\sin\theta'''.1_e$, it is manifest that our conjugate quaternion may be represented by

$$OM_1 = \{-\sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta'' - \sqrt{(-1)}\sin\theta'''\}1_e = 1_e,$$

independent of direction.

12. The rationale of the operation is thus given: Multiplication by a geometrical (that is, a spatial) factor is explained by the transfer of the multiplicand through a distance equal to the extent of the multiplier, such extent being counted from the origin of admeasurement. The result of the multiplication (or value of the product), or of several successive multiplications, is determined by the measure of the distance the multiplicand is finally carried from the origin. If the circumference be taken for spatial multiplier, and the unit radius 1_e loaded with the unit mass 1_μ at its further extremity for multiplicand, the distance of 1_μ from the centre of revolution will be unity, and one complete revolution or multiplication restores it to the same origin and distance from the centre of revolution, and consequently satisfies the equation $1^n.1_e.1_\mu = 1.1_e.1_\mu$. It is obvious, also, that the circumference may be divided into any number

of parts for multipliers, and successive multiplication by all these parts would restore the primitive direction and distance, while each single multiplication would place the unit of mass 1_μ in a position in which its measure on the primitive axis would have a different value.

13. If the unit diagonal OM of a rectangular parallelopipedon, regarded as the measure of a force of the first order, be referred to three rectangular coordinate axes, its projections on these axes will be the measures of its components in their directions, and may be expressed by $+\sqrt{(-1)}\sin\theta'.1_\epsilon$, $+\sqrt{(-1)}\sin\theta''.1_\epsilon$, and $+\sqrt{(-1)}\sin\theta'''.1_\epsilon$; each being equal, in another view, to the revolution of the radius $OM = 1_\epsilon$ through the corresponding angle θ', θ'', θ''', accompanied by a reduction of the value of the coefficient 1 to $\sin\theta'$, $\sin\theta''$, $\sin\theta'''$, so as to satisfy the equation

$$1.1_\epsilon = \{+\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}1_\epsilon.$$

The given quantities are the magnitude of the diagonal, and the angles it makes with the axes of coordinates; and in terms of these data are the magnitudes of the components determined. If now the resultant of the components in the direction of the diagonal be determined, it will give an identical equation; the process consisting in projecting back each component through the angle of its first projection from the diagonal, which is obviously equivalent to effecting the products

$$\{+\sqrt{(-1)}\sin\theta'\}\{-\sqrt{(-1)}\sin\theta'\} = +1\sin^2\theta',$$
$$\{+\sqrt{(-1)}\sin\theta''\}\{-\sqrt{(-1)}\sin\theta''\} = +1\sin^2\theta'',$$
$$\text{and}\quad \{+\sqrt{(-1)}\sin\theta'''\}\{-\sqrt{(-1)}\sin\theta'''\} = +1\sin^2\theta'''.$$

Here the components and their angles with the direction of the resultant are the data, and the magnitude of that resultant is both known and redetermined by the projections through the negative angles $(\frac{1}{2}\pi - \theta')$, $(\frac{1}{2}\pi - \theta'')$, $(\frac{1}{2}\pi - \theta''')$, the cosines of which are $-\sqrt{(-1)}\sin\theta'$, $-\sqrt{(-1)}\sin\theta''$, and $-\sqrt{(-1)}\sin\theta'''$. But revolution through each of these negative angles is evidently equivalent in result to a positive revolution through its complement to the circumference $(2\pi - \theta')$, $(2\pi - \theta'')$, $(2\pi - \theta''')$; and this leads to the effectuation of the product

$$\{+\sqrt{(-1)}\sin\theta' + \sqrt{(-1)}\sin\theta'' + \sqrt{(-1)}\sin\theta'''\}$$
$$\{-\sqrt{(-1)}\sin\theta' - \sqrt{(-1)}\sin\theta'' - \sqrt{(-1)}\sin\theta'''\}$$
$$= (+1\sin^2\theta' + 1\sin^2\theta'' + 1\sin^2\theta''') = 1.$$

14. If, instead of using the angle θ''' counted from the zenithal axis (fig. III.), we write θ for its complement counted from the meridional axis, we fall upon the expression

$$\{\cos\theta+\sqrt{(-1)}\sin\theta'+\sqrt{(-1)}\sin\theta''\}\ \{\cos\theta-\sqrt{(-1)}\sin\theta'-\sqrt{(-1)}\sin\theta''\}$$
$$=\cos^2\theta+\sin^2\theta'+\sin^2\theta''=1,$$

which is the spherical analogue of the more familiar expression

$$\{\cos\theta+\sqrt{(-1)}\sin\theta\}\ \{\cos\theta-\sqrt{(-1)}\sin\theta\}=\cos^2\theta+\sin^2\theta=1$$

for the circle. In operating the multiplications in this case, the same regard to the property of perpendicularity, both between arcs and between axes, must be had as in the case we have preferred to treat on account of its perfect symmetry.

15. The following deduction exhibits the quaternion in a form under which it has been obtained analytically.

Take the diameter M_1M of the sphere, and regard each radius M_1O and OM as positive unity: we can then write

$$\{1+\sqrt{(-1)}\sin\theta'+\sqrt{(-1)}\sin\theta''+\sqrt{(-1)}\sin\theta'''\}1_e$$
$$=M_1O+OP+OQ+OR=M_1O+OM=2.1_e,$$

$$\{1-\sqrt{(-1)}\sin\theta'-\sqrt{(-1)}\sin\theta''-\sqrt{(-1)}\sin\theta'''\}1_e$$
$$=M_1O-OP'-OQ'-OR'=M_1O-OM_1=0.1_e,$$

and the effectuation of the product gives

$$1+\sin^2\theta'+\sin^2\theta''+\sin^2\theta'''=0.$$

Then, if we make the radius $=m_1$, $\sin\theta'=m_2$, $\sin\theta''=m_3$, $\sin\theta'''=m_4$, we get the analytical expression

$$m_1^2+m_2^2+m_3^2+m_4^2=0$$

alluded to, which has also been shown to be the product

$$(m_1+im_2+jm_3+km_4)\ (m_1-im_2-jm_3-km_4)=0,$$

where $i^2=j^2=k^2=-1$; and to these coefficients i, j, k, belongs yet the very remarkable property, that

$$ij=-ji,\quad ik=-ki,\quad \text{and}\quad jk=-kj,$$

which may be accounted for in the following manner.

(1) Take for the primitive quaternion OM any one of the three unit axes, as $OA=+1'.1_e=+i$: then observing

that the operation, which in Sir W. R. Hamilton's lectures is defined to be a linear multiplication by the axis of rotation, is here exhibited as multiplication by a right angle perpendicular to that axis, we may trace the steps as they follow:

$$(+1')\,1_e = \quad +i = OA \ldots\ldots \text{I.}$$
$$(+1')\,\{+\sqrt{(-1)'}\}\,1_e = \quad ik = +j = OB \ldots\ldots \text{I.}$$
$$\{+\sqrt{(-1)'}\}\,\{+\sqrt{(-1)''}\}\,1_e = \quad ji = +k = OC \ldots\ldots \text{II.}$$
$$\{+\sqrt{(-1)'}\}\,(-1'')\,1_e = \quad ki = -j = OB' \ldots\ldots \text{II.}$$
$$\{+\sqrt{(-1)'}\}\,\{-\sqrt{(-1)''}\}\,1_e = -ji = -k = OC' \ldots\ldots \text{II.}$$
$$\{+\sqrt{(-1)'}\}\,(+1'')\,1_e = -ki = +j = OB \ldots\ldots \text{II.}$$
$$(+1'')\,(-1')\,1_e = \quad jk = -i = OA' \ldots\ldots \text{I.}$$
$$(+1'')\,\{-\sqrt{(-1)'}\}\,1_e = -ik = -j = OB' \ldots\ldots \text{I.}$$
$$(+1'')\,(+1')\,1_e = -jk = +i = OA \ldots\ldots \text{I.}$$

(2) Take $OB = +1''.1_e$ for primitive quaternion, and

$$(+1'')\,1_e = \quad +j = OB \ldots \text{II.}$$
$$(+1'')\,\{+\sqrt{(-1)''}\}\,1_e = \quad ji = +k = OC \ldots \text{II.}$$
$$\{+\sqrt{(-1)''}\}\,\{+\sqrt{(-1)'''}\}\,1_e = \quad kj = +i = OA \ldots \text{III.}$$
$$\{+\sqrt{(-1)''}\}\,(-1''')\,1_e = \quad ij = -k = OC' \ldots \text{III.}$$
$$\{+\sqrt{(-1)''}\}\,\{-\sqrt{(-1)'''}\}\,1_e = -kj = -i = OA' \ldots \text{III.}$$
$$\{+\sqrt{(-1)''}\}\,(+1''')\,1_e = -ij = +k = OC \ldots \text{III.}$$
$$(+1''')\,(-1'')\,1_e = \quad ki = -j = OB' \ldots \text{II.}$$
$$(+1''')\,\{-\sqrt{(-1)''}\}\,1_e = -ji = -k = OC' \ldots \text{II.}$$
$$(+1''')\,(+1'')\,1_e = -ki = +j = OB \ldots \text{II.}$$

(3) Lastly, take OC for primitive quaternion, and

$$(+1''')\,1_e = \quad +k = OC \ldots\ldots \text{III.}$$
$$(+1''')\,\{+\sqrt{(-1)'''}\}\,1_e = \quad kj = +i = OA \ldots \text{III}$$
$$\{+\sqrt{(-1)'''}\}\,\{+\sqrt{(-1)'}\}\,1_e = \quad ik = +j = OB \ldots \text{I.}$$
$$\{+\sqrt{(-1)'''}\}\,(-1')\,1_e = \quad jk = -i = OA' \ldots \text{I.}$$
$$\{+\sqrt{(-1)'''}\}\,\{-\sqrt{(-1)'}\}\,1_e = -ik = -j = OB' \ldots \text{I.}$$
$$\{+\sqrt{(-1)'''}\}\,(+1')\,1_e = -jk = +i = OA \ldots \text{I.}$$
$$(+1')\,(-1''')\,1_e = \quad ij = -k = OC' \ldots \text{III.}$$
$$(+1')\,\{-\sqrt{(-1)'''}\}\,1_e = -kj = -i = OA' \ldots \text{III.}$$
$$(+1')\,(+1''')\,1_e = -ij = +k = OC \ldots \text{III.}$$

The diagonal of the cube whose side is unity, is the single quaternion competent to these three processes.

16. Triple products will appear as follow:

(1) $OB = +1.1_e$ ..... II., I.

$OA' = k1_e = \sqrt{(-1)'}.1_e$ ..... I., III.

$OC = kj1_e = \sqrt{(-1)'}.\sqrt{(-1)'''}.1_e$ ..... III., II.

$OB' = kji1_e = \sqrt{(-1)'}.\sqrt{(-1)'''}.\sqrt{(-1)''}.1_e = -1.1_e$...II., I.

(2) $OB = +1.1_e$ ..... II.

$OC = i1_e = \sqrt{(-1)''}.1_e$ ..... II., III.

$OA = ij1_e = \sqrt{(-1)''}.\sqrt{(-1)'''}.1_e$ ..... III., I.

$OB = ijk1_e = \sqrt{(-1)''}.\sqrt{(-1)'''}.\sqrt{(-1)'}.1_e = +1.1_e$...I., II.

(3) $OB' = -1.1_e$ ..... II.

$OC' = i1_e = \sqrt{(-1)''}.1_e$ ..... II., III.

$OA' = ij1_e - \sqrt{(-1)''}.\sqrt{(-1)'''}.1_e$ ..... III., I.

$OB' = ijk1_e = \sqrt{(-1)''}.\sqrt{(-1)'''}.\sqrt{(-1)'}.1_e = -1.1_e$...I., II.

(4) $OB' = -1.1_e$ ..... II., I.

$OA = k1_e = \sqrt{(-1)'}.1_e$ ..... I., III.

$OC' = kj1_e = \sqrt{(-1)'}.\sqrt{(-1)'''}.1_e$ ..... III., II.

$OB = kji1_e = \sqrt{(-1)'}.\sqrt{(-1)'''}.\sqrt{(-1)''}.1_e = +1.1_e$.....II.

Which results show that $ijk = -kji$. (See Hamilton's *Lectures*, p. 208, &c.)

17. In the *London, Edinburgh, and Dublin Philosophical Magazine* for October 1853, p. 283, it is related that Sir W. R. Hamilton has recently employed in some formulæ a fourth imaginary unit h, in addition to i, j, k, appropriated to the three coordinate planes of space; and as this additional imaginary is distinct from the other three, and commutative with them, the author of the article infers that it must be extraspatial. This supposition is not at all necessary: the three symbols of perpendicularity, i, j, k, are restricted each to its *special* plane; while h may hold a *general* value, applicable to each and all of the planes.

N.B. The quaternion has yet other forms than those above given, but here is enough for an example of the

method I have followed in obtaining them. This method arises from carrying out the principles of *the calculus of operations*, as I understand the meaning of the term. Any compound function is the result of certain simple operations, the traces of which are retained in its expression. For instance, the function ψ^n expresses the *abstract* result of n multiplications; and this result will be rendered *concrete* by introducing the unit of space, when it becomes $\psi^n.1_e^n$, or $\psi^n.1_e^n$ if 1_e the linear unit be used. We must begin, then, by inquiring into the result of n operations with *linear* or *angular* unity, or both. When the relation between the diagonal and sides of a rectangular parallelogram, or between the diagonal and sides of a rectangular parallelopipedon, is brought to mind, in which we have

$$1^2_e = (\cos^2\theta + \sin^2\theta)\,1_e = 1_e$$

in the first case, and

$$1^2_e = (\sin^2\theta' + \sin^2\theta'' + \sin^2\theta''')1_e = (\cos^2\theta' + \cos^2\theta'' + \cos^2\theta''')1_e = 1_e$$

in the second, together with the decomposition of each of these compound coefficients of the linear unit into a pair of binomial factors, it is readily perceived that the concrete function of the second degree $\psi^2.1_e^2$ contains values that cannot be evolved from the abstract or merely numerical ψ^2. Attention to the element of space, necessarily involved in the measurement of the result of any operation whatever, has furthermore led to the discovery of the actual genesis of the differential coefficient, and thereby thrown open the entire theory of mathematical development.

THE ATTRACTION OF ELLIPSOIDS CONSIDERED GEOMETRICALLY.

By Matthew Collins, B.A.

The attraction of an ellipsoid A on a point P on its surface or within it, in a direction perpendicular to one of its principal planes B, is proportional to the distance of the attracted point P from that principal plane.

1. When P is on the surface, draw PP', a chord of A, perpendicular to B, and through P and P' draw planes parallel to B, cutting the principal axis CC' perpendicularly in p and p'; then describe through p, p' an ellipsoid a, concentric, similar, and similarly placed to A, and its attraction

on p will be equal to the attraction of A on P in a direction perpendicular to B: for through PP' draw two planes E, F, containing a very small angle, and through pp' draw two planes e, f, parallel to the former; then let a cone of revolution, whose axis is PP' and vertex P, cut E, F along the lines PE, PE', PF, PF'; and let another such cone, very close to the former, also cut E, F along PE_1, PE_1', PF_1, PF_1'; and through p draw in the planes e, f the straight lines pe, pe', pf, pf', and pe_1, pe_1', pf_1, pf_1', respectively parallel to the foregoing: then, as the sections of the two similar ellipsoids A and a, by the parallel planes E, e, are necessarily similar ellipses, and as the chord PP', parallel to an axis (CC') of the greater ellipse, is equal to the homologous axis pp' of the less; therefore, by Airy's *Tract* on the Figure of the Earth, Props. 2, 3,

$$PE + PE' = pe + pe':$$

and therefore, by Airy's 4th Prop., the sum of the attractions exerted on P along PP' by the two small pyramids $PEFE_1F_1$, $PE'F'E_1'F_1'$, is equal to the sum of the attractions exerted on p along pp' by the two corresponding small pyramids $pefe_1f_1$, $pe'f'e_1'f_1'$: and since there are obviously as many pairs of pyramids in the double wedge $PP'EFE'F'$ as there are corresponding pairs of pyramids (whose solid angles are also equal to those of the former) in the double wedge $pp'efe'f'$. And as, moreover, each double wedge of A has a corresponding double wedge of a, therefore the whole attraction of a on p is equal to the attraction of A on P along PP'; but, since a is similar to A, the attraction of a on p is to attraction of A on C as pp' ($= PP'$) to CC' (*Principia*, Prop. 87, Cor. 1); and so the attraction of A on P perpendicular to B, which was proved equal to the attraction of a on p, is therefore $= \frac{\frac{1}{2}PP'}{\frac{1}{2}CC'} \times$ attraction of A on C; which, since A and CC' are constant, $\propto \frac{1}{2}PP'$, viz. the distance of P from B.

The general equation of a surface of the second order being

$$Q(xyz) = A + Bx + Cy + Dz + Ex^2 + \&c. = 0,$$

the equation of the diametral plane bisecting all chords parallel to the straight line $x = mz$ and $y = nz$, is known to be

$$md_xQ + nd_yQ + d_zQ = 0,$$

which, not containing the absolute constant term A, shews

that if *any* straight line $ABB'A'$ cuts two surfaces of the second order, whose equations differ only in the constant term, the intercepts AB, $A'B'$ will be equal, since the chords AA', BB' are bisected in the same point by the diametral plane conjugate to it, which plane is the same for both surfaces. Now, the equations of two ellipsoids, which are concentric, similar and similarly placed, differ only in the absolute constant term, and therefore the intercepts of *any* straight line cutting two such ellipsoids are equal; and hence it follows that a shell, bounded by concentric, similar and similarly placed ellipsoidal surfaces, exerts no attraction on a point P situated anywhere within it, or upon its interior surface. See Airy's *Tract* on the Figure of the Earth, Prop. 12, or the *Principia*, Props. 70 and 91, Cor. 3, Book 1.

(2) When P is within the ellipsoid A, we have then only to describe through P another ellipsoid A' concentric, similar and similarly placed to A; and since the shell between A and A' exerts no attraction on P, as is already proved, therefore the attraction of A on P is the same as that of A' on P; then, as in (1), draw PP' a chord of A' perpendicular to B, and through P and P' draw planes parallel to B, cutting the principal axis CC' perpendicularly in p and p', and then describe through p and p' another ellipsoid a, concentric, similar and similarly situated to A or A': and by (1), the attraction of A (or A') on P perpendicularly to B is equal to the whole attraction of a on p, and therefore $= \frac{\frac{1}{2}PP'}{\frac{1}{2}CC'} \times$ attraction of A on C, which, as already observed, $\propto \frac{1}{2}PP'$, viz. the distance of P (now supposed within A) from B.

[The foregoing elementary geometrical demonstration is that alluded to and promised in the Note at the bottom of page 51 of the No. of this Journal for February 1854.]

The preceding proposition shews that the attraction of an ellipsoid on any point on its surface or within it, can be got at once from the attraction of the ellipsoid on a point placed at the extremity of an axis; and this latter attraction is found, and reduced to elliptic functions, as follows.

Let O be the centre and $OA = a$, $OB = b$, $OC = c$, the semiaxes; and let the attracted point C be the vertex, and CO the axis of a cone of revolution D whose semiangle is θ, and let $\theta + d\theta$ be the semiangle of another such cone E,

very close to D, and having the same vertex and axis; and let O be the common vertex of two other cones D' and E' parallel to the former. Conceive the portion of the ellipsoid between D and E to be divided into elementary pyramids by planes passing through CO: let f be the length of a side of one of these little pyramids, which is a chord of the ellipsoid and a side of D, and let g be the parallel side of D', which is a radius of the ellipsoid; and let f' and g' be the projections of f and g upon c, and let ω be the small angle between two consecutive f's (or g's); then the attraction of the little pyramid, whose side is f, on its vertex C is equal to $f\omega d\theta$, (vide Airy's *Tract* on the Figure of the Earth, Prop. 4), and therefore its component along CO is equal to $f'\omega d\theta$, therefore

$$= 2c \times \frac{f'}{2c}\,\omega d\theta;$$

but $\frac{f'}{2c} = \frac{g'^2}{c^2}$, therefore the said component $= \frac{2}{c}\,g'^2\omega d\theta$, therefore

$$= \frac{4}{c} \times \frac{1}{2}\,g^2\cos^2\theta\omega d\theta.$$

Now $\frac{1}{2}g^2\omega$ is the area on D' included between the two consecutive g's, and the sum of all such elements is the entire surface of D', which we shall still name D'; therefore the attraction of the slice of the ellipsoid between D and E upon C along CO is $= \frac{4D'\cos^2\theta d\theta}{c}$. Now the projection of D' on the plane of ab is obviously an ellipse, whose area D'' is $= D'\sin\theta$. Let now r and r' be the sides of D' (or radii of the ellipsoid) lying in the planes of ca and cb, then the semiaxes of D'' are plainly the projections of r and r', or $r\sin\theta$ and $r'\sin\theta$, and therefore $D'' = \pi rr'\sin^2\theta$; and so $D' = \frac{D''}{\sin\theta}$, therefore $= \pi rr'\sin\theta$: and so the attraction of the slice on C along CO is $= \frac{4\pi}{c}\,rr'\cos^2\theta\sin\theta d\theta$. Now $\frac{1}{r^2} = \frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta}{a^2}$ and $\frac{1}{r'^2} = \frac{\cos^2\theta}{c^2} + \frac{\sin^2\theta}{b^2}$, and therefore the differential of the required attraction on C is

$$= \frac{4\pi abc\cos^2\theta\sin\theta d\theta}{(c^2\sin^2\theta + a^2\cos^2\theta)^{\frac{1}{2}}(c^2\sin^2\theta + b^2\cos^2\theta)^{\frac{1}{2}}}.$$

This expression, given in Poisson's *Mécanique*, vol. I. p. 190,

by putting $\cos\theta = u = (1+v)^{-\frac{1}{2}}$, becomes transformed into

$$\frac{4\pi abc u^2 du}{\{c^2+(a^2-c^2)u^2\}^{\frac{1}{2}}\{c^2+(b^2-c^2)u^2\}^{\frac{1}{2}}} = \frac{2\pi c\, dv}{(1+v)\sqrt{\left\{(1+v)\left(1+\frac{c^2}{a^2}v\right)\left(1+\frac{c^2}{b^2}v\right)\right\}}},$$

where $v = \tan^2\theta$.

Now, supposing $a > b > c$, let OA' and OB', portions of OA and OB, be the semiaxes of the focal ellipse whose plane is perpendicular to the least semiaxis OC, then

$$OA'^2 = a^2 - c^2 \quad \text{and} \quad OB'^2 = b^2 - c^2;$$

let OQ be perpendicular to the tangent PTQ which touches this ellipse in T and meets OA' at the point P, and let the angle $OPT = \phi$, and $\rho^2 = c^2 + (b^2-c^2)u^2 = c^2 + OB'^2\cos^2\theta$, and $\rho'^2 = c^2 + (a^2-c^2)u^2 = c^2 + OA'^2\cos^2\theta$; and if the point P be taken such that $\frac{OP}{OA'} = \frac{\rho}{c}$ or $OP = \frac{OA'}{c}\rho$, then the equation $OA'^2\cos^2A'OQ + OB'^2\cos^2B'OQ = OQ^2$, $\therefore = OP^2\sin^2\phi$, gives

$$(a^2-c^2)\sin^2\phi + (b^2-c^2)\cos^2\phi = \frac{a^2-c^2}{c^2}\{c^2+(b^2-c^2).\cos^2\theta\}\sin^2\phi,$$

which gives

$$(a^2-c^2)\cos^2\theta = \cos^2\phi\{c^2+(a^2-c^2).\cos^2\theta\},$$

i.e. $OA'^2\cos^2\theta = \rho'^2\cos^2\phi$, and therefore $\cos\phi = \frac{OA'}{\rho'}\cos\theta$;

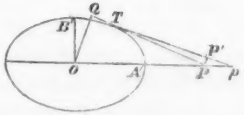
and since $OP = \frac{OA'}{c}\rho = \frac{OA'}{c}(c^2+OB'^2\cos^2\theta)^{\frac{1}{2}}$,

therefore

$$Pp = d.OP = \frac{OA'}{c}(c^2+OB'^2\cos^2\theta)^{-\frac{1}{2}}.OB'^2\cos\theta\sin\theta.d\theta$$

$$= \frac{OA'.OB'^2}{c\rho}\cos\theta\sin\theta.d\theta.$$

Let PP' be perpendicular to the consecutive tangent from p, then $pP' = Pp\cos\phi$, therefore



$$= \frac{OA'^2.OB'^2}{c\rho'}\cos^2\theta\sin\theta.d\theta,$$

but the attraction of the slice of the ellipsoid between D and E on C along CO was found above

$$= \frac{4\pi abc\cos^2\theta\sin\theta.d\theta}{\rho\rho'}.$$

And moreover, since $P'p$ is obviously $= d.B'TP$, therefore

$$= d(B'TP - \text{const. } B'TA'),\ \textit{i.e.}\ = d(TP - \text{arc } TA');$$

therefore the attraction of the slice

$$= \frac{4\pi abc^2}{OA'^2.OB'^2} \times P'p,\ \textit{i.e.} = \frac{4\pi abc^2}{(a^2-c^2)(b^2-c^2)}\, d(TP - \text{arc } TA');$$

and therefore the attraction of the whole ellipsoid on C is

$$= \frac{4\pi abc^2}{(a^2-c^2)(b^2-c^2)} \times (T_1P_1 - \text{arc } T_1A'),$$

where P_1 and T_1 denote the ultimate positions of P and T corresponding to $\theta = 0$; and since, by construction,

$$OP = \frac{OA'}{c}\rho = \frac{OA'}{c}\{c^2 + (b^2-c^2).\cos^2\theta\};$$

therefore, when $\theta = 0$, $OP_1 = \frac{b}{c}OA'$, and therefore

$$P_1T_1 = \frac{a(b^2-c^2)}{bc} = \frac{a^2(b^2-c^2)}{abc}.$$

Hence also the differential of the ellipsoid's attraction on B, *i.e.* the attraction on B along BO of a portion of the ellipsoid comprised between two cones of revolution whose vertex is B and axis BO, and semiangles θ and $\theta + d\theta$, is

$$= \frac{4\pi abc\cos^2\theta\sin\theta.d\theta}{(b^2\sin^2\theta + c^2\cos^2\theta)^{\frac{1}{2}}(b^2\sin^2\theta + a^2\cos^2\theta)^{\frac{1}{2}}}$$

$$= \frac{4\pi abcu^2du}{\{b^2-(b^2-c^2).u^2\}^{\frac{1}{2}}\{b^2+(a^2-b^2).u^2\}^{\frac{1}{2}}},$$

where, as before, $u = \cos\theta$. Now, to represent this geometrically, let OA'' and OC', portions of OA, OC, be the semiaxes of the focal hyperbola whose plane is perpendicular to OB, then

$$OA''^2 = a^2 - b^2 \text{ and } OC'^2 = c^2 - b^2 = -(b^2-c^2) = -OB'^2;$$

and putting now

$$\rho^2 = b^2 - (b^2-c^2)\cos^2\theta \text{ and } \rho'^2 = b^2 + (a^2-b^2)\cos^2\theta,$$

and taking the point P on the primary axis OA'' so that $\frac{OP}{OA''} = \frac{\rho}{b}$, so that, as $\rho < b$, P will lie between O and A''; then, drawing PT touching the hyperbola, we find, as before, *mutatis mutandis*, the whole of the attraction on B

$$= \frac{4\pi ab^3c}{(a^2 - b^2)(b^2 - c^2)} (T_1P_1 - \text{arc } T_1A''),$$

P_1 and T_1 being now, as before, the ultimate positions of P and T corresponding to $\theta = 0$, so that, as before,

$$OP_1 = \frac{c}{b} OA''.$$

The whole attraction on A cannot be represented similarly, because there is no focal conic perpendicular to OA; but the equation $\frac{A}{a} + \frac{B}{b} + \frac{C}{c} = 4\pi$, discovered by the late Professor M'Cullagh, will then serve to A, where A, B, C denote the whole attractions of the ellipsoid on the points A, B, C.

Let a, b, c, be the semiaxes of an homogeneous fluid ellipsoid, and A, B, C the attractions on points at the ends of a, b, c, caused partly by the ellipsoid's own attractions on its parts, partly by centrifugal force of revolution about an axis ($2c$), or by the action of an extraneous force directed towards the centre, and varying as the distance from the centre; then the ellipsoid will preserve its shape if $Aa = Bb = Cc$.

For then the whole forces, acting on any particle xyz of the mass in directions parallel to a, b, c, will obviously be

$$\frac{Ax}{a}, \quad \frac{By}{b}, \quad \text{and} \quad \frac{Cz}{c}.$$

Now, dividing these by $Aa = Bb = Cc$, they are as

$$\frac{x}{a^2}, \quad \frac{y}{b^2}, \quad \text{and} \quad \frac{z}{c^2};$$

but when the point xyz is on the surface, these last are as the cosines of the angles that the normal at the point xyz makes with the axes, since the equation of the tangent plane is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1.$$

Thus the components of the force on the point xyz at the surface are as the cosines of the angles that the normal at this point makes with the axes; therefore the direction of the resultant force is coincident with the normal, or perpendicular to the surface itself, which is the necessary condition of equilibrium, the general formula

$$dp = \rho(Xdx + Ydy + Zdz)$$

obviously becomes in this case

$$dp = -\rho\left(\frac{Ax}{a}\,dz + \frac{By}{b}\,dy + \frac{Cz}{c}\,dz\right)$$

$$= -\frac{\rho Aa}{2}\left(\frac{2x\,dx}{a^2} + \frac{2y\,dy}{b^2} + \frac{2z\,dz}{c^2}\right),$$

and therefore $p = C' - \frac{\rho Aa}{2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)$ at the surface $p = 0$, and $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, therefore $C' = \frac{\rho Aa}{2}$, and therefore the pressure $= \frac{\rho Aa}{2}\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)$, which, at the centre, where $x = y = z = 0$, becomes $\frac{\rho Aa}{2}$. (See the *Principia*, Prop. 20, Book III.; M'Laurin's *De Causa Phys. Flux et Reflux*, Maris, Prop. 1; Airy's *Tract* on the Figure of the Earth, Props. 14, 15, 16.)

Let R and r be the radii of two homogeneous concentric globes, A and a the attractions of each on a point on the surface of the other, then $\frac{A}{R^2} = \frac{a}{r^2}$, whatever be the law of attraction as a function of the distance.

Let O be their common centre, OrR a radius meeting them at r and R, bc a chord of less parallel to OR: produce Ob, Oc to meet the large globe at B, C, then BC will be parallel to bc or to OR; and if b describe any little figure b' on the surface of r, it is evident B will describe a similar figure B' on the surface of R, and the areas S, s of the normal sections of the cylinders C and c, simultaneously described by bc and BC, will obviously be to each other as $B' : b'$, therefore as $R^2 : r^2$. Now, by Euclid, Prop. 4, Book I., $Br = bR$ and $Cr = cR$; therefore, by Airy's *Tract* on the Figure of the Earth, Prop. 18 (generalized), attraction of cylinder C on the point r along rO : attraction of cylinder c on R along $RO :: S : s :: R^2 : r^2$; and as this

fixed proportion holds true for each corresponding pair of cylinders, therefore by taking their sums we shall also have $A : a :: R^2 : r^2$. (See Poisson's *Mécanique*, vol. 1. p. 201.)

P.S.—Several of the propositions given in the present paper, and in that I have inserted in the February No. of this Journal, were given in 1846 by the late Professor M'Cullagh at his lectures. I have, however, departed in several places from his method of proof, and supplied some connecting links which seemed most necessary.

Mechanics' Institute, Liverpool, April 15, 1854.

ON THE AREA OF THE CYCLOID.

To the Editor of the Cambridge and Dublin Mathematical Journal.

Sir,—The determination of the area of the Cycloid, so easily effected by modern analysis, was regarded by the geometricians of the seventeenth century as a problem of no small difficulty. Mersenne was the first who attempted a solution: he was however unsuccessful. It was proposed by him in despair to Roberval in 1628, who also failed in his attempt at that time. About seven years afterwards, however, Roberval overcame the difficulty and communicated his good fortune to Mersenne.

In a letter to Descartes, Mersenne made mention of Roberval's discovery of the area of the Cycloid as a great feat in geometry, simply stating the result obtained by Roberval, without giving any clue to the method. Descartes, solving the problem himself with little difficulty, communicated his method in reply to Mersenne, with some supercilious remarks about the supposed difficulty of the problem. Fermat and other mathematicians of that day exercised their ingenuity in the same question. A solution of the problem by pure geometry, which was some time ago communicated to me by Mr. R. L. Ellis of Trinity College, possesses so great a superiority over any of the geometrical methods of these early mathematicians which I have seen, that I think it may be acceptable to those readers of your Journal who take an interest in the history of mathematics.

"The motion of the generating circle may be resolved into two uniform motions, a motion of translation parallel to the directrix and of rotation round its own centre. The area generated by the describing point may be considered as generated by these two motions: that of translation nowise

affects the motion of rotation, and the area due to the latter is the same as if the former did not exist, that is, it is equal to the area of the generating circle. Contrariwise the motion of rotation does affect the area due to that of translation, inasmuch as in virtue of it the distance of the describing point from the directrix is varied: the mean distance, viewed as depending on the motion of rotation, is equal to the radius of the generating circle, and the corresponding area is therefore a rectangle, the base of which is the space slided over and altitude that radius; and, as this space is the circumference of the generating circle, the area in question is equal to twice the area of that circle: on the whole, therefore, the area of the cycloid is equal to three times that of the generating circle.

"The reason is just the same as that by which what are called Guldinus's properties are established. We here resolve the motion of a describing point into motions parallel and perpendicular to the abscissa; the latter generates no area, the former generates a rectangular area having for its base the abscissa and for its altitude the mean value of the ordinates; that is, the ordinate of the centre of gravity of the arc, which is a known result. The only difference to be attended to in the two cases relates to the mode in which the average is to be taken."

Mr. Ellis has remarked, that the same method may be extended to the determination of the areas of the hypocycloid and epicycloid.

I am, Sir,

Your obedient Servant,

WILLIAM WALTON.

Cambridge, July 31, 1854.

PROOF OF THE PARALLELOGRAM OF FORCES.

By Arthur Cohen.

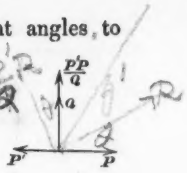
The only difficulty in the analytical proof of the parallelogram of forces consists in determining the direction of the resultant of two forces acting at right angles to one another.

This problem may, in the following manner, be reduced to the solution of the functional equation

$$\phi(x) + \phi(y) = \phi(x+y),$$

which is easily solved by ordinary algebra.

Let P and Q be two forces acting at right angles to one another, R their resultant inclined to P at an angle θ. The direction of R and its ratio to P clearly depend only on the ratio $\frac{P}{Q}$. We may therefore put



$$\theta = \phi(\alpha) \text{ where } \tan(\alpha) = \frac{Q}{P}.$$

In direction of Q and in direction opposite to P apply respectively two forces $\frac{P'P}{Q}$ and P', and let R' be their resultant; then, since $P : Q :: \frac{P'P}{Q} : P'$, it is clear that R' and R are at right angles to one another. Let the resultant of R' and R be inclined to R at an angle θ'. Then

$$\theta' = \phi(\alpha') \text{ where } \tan(\alpha') = \frac{R'}{R} = \frac{P'}{Q};$$

and since R' is also the resultant of the forces $P - P'$, $Q + \frac{P'P}{Q}$, therefore

$$\theta + \theta' = \phi(\beta) \text{ where } \tan(\beta) = \frac{Q + \frac{P'P}{Q}}{P - P'} = \frac{\frac{Q}{P} + \frac{P'}{Q}}{1 - \frac{P'}{Q} \cdot \frac{Q}{P}} = \tan(\alpha + \alpha');$$

therefore $$\beta = \alpha + \alpha';$$

therefore $$\phi(\alpha) + \phi(\alpha') = \phi(\alpha + \alpha'):$$

the solution of which last equation gives $\phi(\alpha) = C\alpha$, C being a constant, that must equal unity, inasmuch as when

$$P = Q \tan(\theta) = 1 \text{ and } \tan(\alpha) = 1;$$

therefore $\theta = \alpha$, or the direction of the resultant coincides with that of the diagonal of the rectangle constructed on the two component forces.

THEOREMS ON THE QUADRATURE OF SURFACES AND THE RECTIFICATION OF CURVES.

By Rev. Robert Carmichael, A.M.,

Fellow of Trinity College, Dublin, and Examiner in Mathematics for the year 1854-5 in the Queen's University in Ireland.

1. It is well known that there are many plane curves whose equations are more easily expressed in polar than in rectangular coordinates, and for whose rectification we employ the formula

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\{r^2 + (D_\theta r)^2\}}\, d\theta.$$

Of this class are, the Spiral of Archimedes,

$$r = a\theta;$$

the Lituus,

$$r^2 = \left(\frac{a}{\theta}\right)^2;$$

the Lemniscate,

$$r^2 = a^2 \cos 2\theta;$$

the Logarithmic Spiral,

$$r = ce^{\frac{\theta}{a}};$$

and the Cardioid,

$$r = a(1 - \cos\theta).$$

2. I am not aware that any mathematician has attempted to trace the surfaces analogous to these; but, for the *quadrature* of such surfaces, when discovered, it is absolutely necessary that we should have a general expression *in polar coordinates* for the element of any surface. Such an expression is not found in the ordinary works upon the Differential and Integral Calculus. In the elaborate treatise upon this subject by M. L'Abbé Moigno (*Paris*, 1844, tom. II. p. 235), the expression is investigated, by the usual analytical method, transformation of coordinates, from the well-known expression in rectangular coordinates,

$$d\sigma = \sqrt{(1 + p^2 + q^2)}\, dx\, dy,$$

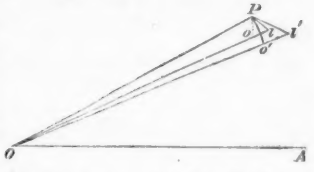
and is given in the following shape,

$$d\sigma = \sqrt{\{r^2 \sin^2\theta + \sin^2\theta (D_\theta r)^2 + (D_\phi r)^2\}}\; r\, d\theta\, d\phi.$$

A short geometrical deduction of this expression, whose merit I have great pleasure in sharing with my friend,

Alexander Jack, Esq., A.B., may not be unacceptable to the student.

Let P be any point on the surface. Through the axis OA and OP describe a plane, and round the axis describe, with the same line, a cone. The surface may then be supposed to be divided into its elements by planes and cones consecutive to these respectively, (the planes all passing through the axis and the cones round it,) half of one such element being represented by $\iota P\iota'$. Then, remembering that the planes cut the cones orthogonally, we have



$$d\sigma = P\iota.P\iota'.\sin \iota P\iota' = P\iota.P\iota'.\sqrt{(1 - \cos^2 \iota P\iota')},$$

whence

$$d\sigma = P\iota.P\iota'.\sqrt{(1 - \sin^2 \iota Po.\sin^2 \iota' Po')} = \sqrt{(P\iota^2.P\iota'^2 - o\iota^2.o'\iota'^2)},$$

o and o' being the points where the sphere described round the origin with radius OP intersects the consecutive radii vectores to the points ι, ι'; or

$$d\sigma = \sqrt{[\{r^2 \sin^2\theta\, d\phi^2 + (D_\phi r)^2 d\phi^2\}.\{r^2 d\theta^2 + (D_\theta r)^2 d\theta^2\} - (D_\theta r)^2 d\theta^2.(D_\phi r)^2 d\phi^2]},$$

or finally,

$$d\sigma = \sqrt{\{r^2 \sin^2\theta + \sin^2\theta\,(D_\theta r)^2 + (D_\phi r)^2\}}\; r\, d\theta\, d\phi.$$

3. From this expression we may readily derive that for the perpendicular from the origin upon the tangent plane, in *polar* coordinates. In *rectangular* coordinates it is known to be

$$P = \frac{z - px - qy}{\sqrt{(1 + p^2 + q^2)}};$$

but the transformation of this to polar coordinates would be troublesome and tedious. We may easily derive the required expression from the volume of the elementary cone, for

$$P d\sigma = r^3 \sin\theta\, d\theta\, d\phi,$$

and therefore

$$P = \frac{r^3 \sin\theta}{\sqrt{\{r^2 \sin^2\theta + \sin^2\theta (D_\theta r)^2 + (D_\phi r)^2\}}}.$$

4. As an example of the application of the formula for the quadrature of surfaces, let us suppose that it is re-

quired to investigate the quadrature, between given limits, of the surface

$$r = me^{-\phi}\cos\theta.$$

Then

$$D_\theta r = -me^{-\phi}\sin\theta, \quad D_\phi r = -m\,e^{-\phi}\cos\theta;$$

therefore

$$d\sigma = \sqrt{(m^2e^{-2\phi}\cos^2\theta.\sin^2\theta + m^2e^{-2\phi}.\sin^4\theta + m^2e^{-2\phi}\cos^2\theta)}\, r\,d\theta\,d\phi,$$

or

$$d\sigma = m^2e^{-2\phi}\cos\theta\,d\theta\,d\phi;$$

whence

$$\Sigma = m^2\int e^{-2\phi}(\sin\theta_2 - \sin\theta_1)\,d\phi.$$

Let us suppose the limits to be given by the intersections, with the given surface, of the cones

$$\theta_2 = a\phi, \quad \theta_1 = b\phi,$$

and

$$\Sigma = m^2\int_{\phi_1}^{\phi_2} e^{-2\phi}(\sin a\phi - \sin b\phi)\,d\phi,$$

an integral which is susceptible of easy reduction, since we know that

$$\int e^{-m\phi}\sin a\phi\,d\phi = -e^{-m\phi}\,\frac{m\sin a\phi + a\cos a\phi}{m^2 + a^2}.$$

5. As a second example, let it be proposed to investigate the quadrature, within given limits, of the surface

$$r = m\cos\phi\sin\theta.$$

Here

$$D_\theta r = m\cos\phi\cos\theta, \quad D_\phi r = -m\sin\phi\sin\theta,$$

and

$$d\sigma = m^2\cos\phi\sin^2\theta\,d\theta\,d\phi;$$

whence

$$\Sigma = m^2\int_{\theta_1}^{\theta_2}(\sin\phi_2 - \sin\phi_1)\sin^2\theta\,d\theta;$$

and, if the limits be given as before, there is no difficulty in determining the quadrature completely.

6. In the masterly treatise upon the Calculus of Variations by the Rev. Professor Jellett (*Dublin*, 1850, p. 262), it is shown that the surface which, within given limits, renders the double integral

$$\iint\sqrt{(p^2 + q^2)}\,dx\,dy,$$

or, γ being the angle made by the radius vector with the axis of z,

$$\iint \sin\gamma . d\sigma,$$

a minimum, is given by the partial differential equation

$$q^2 r - 2pqs + p^2 t = 0,$$

whose integral is known to be

$$y = xF_1(z) + F_2(z),$$

representing the gauche surface generated by a line which, gliding upon two fixed directrices, remains constantly parallel to a given plane; as indeed might be anticipated from a consideration of the question in its second form.

In the same manner it might be shewn that the surface which, within given limits, renders the double integral

$$\iint \sqrt{\{(D_\theta r)^2 + (D_\phi r)^2\}}\, d\theta\, d\phi$$

a minimum, is given by the equation

$$\phi = \theta F_1(r) + F_2(r).$$

If it be proposed to investigate the property of this surface corresponding to the character of the generation of the analogous surface in rectangular coordinates, as the latter character is exhibited by the supposition $z = const.$, so the former property may be investigated by the supposition $r = const.$ Let then the surface be supposed to intersect a sphere described round the origin, and let the nature of the curve of intersection be examined. If we resolve any element into its rectangular components, one such component is $r\,d\theta$ and the other $r \sin\theta\, d\phi$. Let i be the inclination of the element to the meridional plane described through its extremity and the fixed axis, and it is evident that

$$\tan i = \frac{r \sin\theta\, d\phi}{r\, d\theta} = F_1(c) \sin\theta,$$

c being the radius of the sphere; or *the tangent of the angle of inclination of the curve to the meridional plane is proportional to the sine of the angle made by the radius vector with the axis.*

7. It may be well here to indicate certain desiderata, the knowledge of which might lead to the discovery of some interesting properties of surfaces.

The *measure of curvature* at any point of a surface is expressed in rectangular coordinates by the formula

$$\frac{1}{R_1 R_2} = \frac{rt - s^2}{(1 + p^2 + q^2)^{\frac{1}{2}}};$$

we have no corresponding expression in polar coordinates. Such might be discovered by the investigation of the analogue of the known formula for plane curves

$$\rho = r \frac{dr}{dp}.$$

Again, the *sum of the curvatures* at any point of a surface is expressed by the formula, in rectangular coordinates,

$$\frac{1}{R_1} + \frac{1}{R_2} = -\frac{(1 + q^2) r - 2pqs + (1 + p^2) t}{(1 + p^2 + q^2)^{\frac{3}{2}}};$$

we have no corresponding expression in polar coordinates. Other desiderata will readily suggest themselves.

8. With regard to the rectification of curves, it may be useful to make a few observations upon a subject which has recently attracted much attention among French mathematicians. In the Notes by M. Liouville to his valuable edition of the *Application de l'Analyse à la Géometrie* of the illustrious Monge, will be found (p. 558) the following remarks.

"M. Serret a fait usage de certaines variables qu' il avait déjà employées au tome XIII. du *Journal de Mathématiques*, pour resoudre le problème suivant: x, y, z, s, étant quatre fonctions d'une variable indépendente θ assujetties a verifier l'équation

$$dx^2 + dy^2 + dz^2 = ds^2,$$

exprimer sans forme finie et sans aucun signe d'intégration, les valeurs générales de ces fonctions. La solution de ce problème conduit, par exemple, a trouver des courbes à double courbure qui soient à la fois algébriques et rectifiables algébriquement, ou dont l'arc dépende d'une transcendante donnée. Le problème analogue pour les courbes planes dépend de l'équation plus simple

$$dx^2 + dy^2 = ds^2,$$

et se resont, comme on sait, par les formules

$$x = \psi'(\theta)\sin\theta + \psi''(\theta)\cos\theta,$$

$$y = \psi'(\theta)\cos\theta - \psi''(\theta)\sin\theta,$$

$$s = \psi(\theta) + \psi''(\theta),$$

ou la fonction ψ est arbitraire. Les formules de M. Serret pour l'équation

$$dx^2 + dy^2 + dz^2 = ds^2,$$

sont beaucoup plus compliquées, et, partant, beaucoup moins utiles."

It appears to me that the integration of these equations may be effected directly, and with great simplicity, by employing the Calculus of Quaternions.

Thus, in the notation of this Calculus, the first equation

$$dx^2 + dy^2 = ds^2$$

is equivalent to

$$-(i\,dx + j\,dy)^2 = -(d\rho)^2,$$

or

$$i\,dx + j\,dy = d\rho;$$

whence

$$ix + jy = \rho + \alpha,$$

α being an arbitrary vector; or, between given limits,

$$i(x_2 - x_1) + j(y_2 - y_1) = \rho_2 - \rho_1,$$

an identity, as it ought to be.

Similarly, the second equation

$$dx^2 + dy^2 + dz^2 = ds^2$$

is equivalent to

$$-(i\,dx + j\,dy + k\,dz)^2 = -(d\rho)^2$$

or

$$i\,dx + j\,dy + k\,dz = d\rho;$$

whence

$$ix + jy + kz = \rho + \alpha,$$

α being an arbitrary vector, or, between given limits,

$$i(x_2 - x_1) + j(y_2 - y_1) + k(z_2 - z_1) = \rho_2 - \rho_1,$$

an identity, as it ought to be.

5, Trinity College, Dublin, Nov. 1854.

ON THE INTEGRATION OF LINEAR AND PARTIAL DIFFERENTIAL EQUATIONS.

By ARTHUR H. CURTIS, A.M.
Trinity College, Dublin.

IN a paper published in the No. of the *Cambridge and Dublin Mathematical Journal* for November 1851, Mr. Carmichael has given a method for integrating partial differential equations of a certain class. This method rests on the properties of a symbol of operation, which Mr. Carmichael has termed the Index Symbol. I propose in the first part of this paper to shew, that for this symbol we may substitute a more general one, which may be employed in the integration of a wider class of equations. In the second part I propose to consider an equation which has been integrated by Dr. Hargreave in the *Philosophical Transactions* for 1848, and to deduce its solution in a form more readily interpretable.

I. Mr. Carmichael employs the following symbolic equations:—

$$\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}+\&c...+\alpha\right)^{-1}0=u_{-\alpha}$$

$$\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}+\&c...+\alpha\right)^{-m}0=u_{-\alpha}(\log x+\log y+\log z+\&c.)^{m-1}+v_{-\alpha}(\log x+\log y+\log z+\&c.)^{m-2}+\&c.\ldots\ldots\ldots+w_{-\alpha}$$

$$\left(x\frac{d}{dx}+y\frac{d}{dy}+z\frac{d}{dz}+\&c...+\alpha\right)^{-1}f_{-\alpha}=\frac{1}{n}(\log x+\log y+\log z+\&c.)f_{-\alpha} \quad \ldots\ldots(1),$$

where $u_{-\alpha}$, $v_{-\alpha}$, &c. are *arbitrary* homogeneous functions of the degree $-\alpha$ of the variables x, y, z, &c.; $f_{-\alpha}$ a *given* function of the same character; and n the number of variables.

Now, if we consider that

$$\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c.\ldots.+\alpha\right)=\left(x^{\frac{1}{a}}\frac{d}{dx^{\frac{1}{a}}}+y^{\frac{1}{b}}\frac{d}{dy^{\frac{1}{b}}}+z^{\frac{1}{c}}\frac{d}{dz^{\frac{1}{c}}}+\&c.\ldots.+\alpha\right),$$

it is evident that by changing x, y, z, &c. in these equations

into $x^{\frac{1}{a}}$, $y^{\frac{1}{b}}$, $z^{\frac{1}{c}}$, &c. respectively, we obtain the following more general ones:—

$$\left.\begin{aligned}
&\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c...+\alpha\right)^{-1}0=u_{-\alpha}\\
&\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c...+\alpha\right)^{-m}0=u_{-\alpha}(\log x^{\frac{1}{a}}+\log y^{\frac{1}{b}}+\log z^{\frac{1}{c}}+\&c.)^{m-1}\\
&\qquad\qquad+v_{-\alpha}(\log x^{\frac{1}{a}}+\log y^{\frac{1}{b}}+\log z^{\frac{1}{c}}+\&c.)^{m-2}\\
&\qquad\qquad+\&c.\ldots\ldots\ldots\ldots+w_{-\alpha}\\
&\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c...+\alpha\right)^{-1}f_{-\alpha}\\
&\qquad\qquad=\frac{1}{n}(\log x^{\frac{1}{a}}+\log y^{\frac{1}{b}}+\log z^{\frac{1}{c}}+\&c...)f_{-\alpha}
\end{aligned}\right\}\quad\ldots.(2),$$

where $u_{-\alpha}$, $v_{-\alpha}$, &c. are arbitrary homogeneous functions of the degree $-\alpha$ of $x^{\frac{1}{a}}$, $y^{\frac{1}{b}}$, $z^{\frac{1}{c}}$, &c.; $f_{-\alpha}$ a given function of the same nature; and n the number of variables.*

The class of equations immediately integrable by these symbolic formulæ is represented by the equation

$$\left\{\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c.\ldots.+\alpha\right)\left(a'x\frac{d}{dx}+b'y\frac{d}{dy}+c'z\frac{d}{dz}+\&c...+\alpha'\right)\ldots\right\}u=V\ldots(3),$$

where V is a function of x, y, z, &c., which may be supposed to be of the form $\Sigma Ax^my^nz^r\ldots\ldots$

The inversion of the operator involved in (3) gives

$$u=\frac{1}{\left(ax\frac{d}{dx}+by\frac{d}{dy}+\&c...+\alpha\right)\left(a'x\frac{d}{dx}+b'y\frac{d}{dy}+\&c...+\alpha'\right)\ldots}\Sigma Ax^my^nz^r\ldots$$
$$+\frac{L}{\left(ax\frac{d}{dx}+by\frac{d}{dy}+\ldots+\alpha\right)}0+\frac{M}{\left(a'x\frac{d}{dx}+b'y\frac{d}{dy}+\ldots+\alpha'\right)}0+\&c.$$

* The process by which I first inverted the symbol

$$\left(ax\frac{d}{dx}+by\frac{d}{dy}+cz\frac{d}{dz}+\&c.\ldots.+\alpha\right),$$

was more artificial than that here employed, but the result once obtained, its form suggested to me the proof given above.

where L, M, N, &c. are determinate constants, each of which may be changed into unity, as

$$\frac{L}{\left(ax\frac{d}{dx}+by\frac{d}{dy}+\&c\ldots a\right)}0=\frac{1}{\left(ax\frac{d}{dx}+by\frac{d}{dy}\ldots\right)}L0$$

$$=\frac{1}{\left(ax\frac{d}{dx}+by\frac{d}{dy}\ldots\right)}0.$$

Now $$\left(x\frac{d}{dx}\right)^s Ax^my^nz^r\ldots=(m)^sAx^my^nz^r\ldots,$$

$$\left(y\frac{d}{dy}\right)^{s'}Ax^my^nz^r\ldots=(n)^{s'}Ax^my^nz^r\ldots,$$

or generally

$$\phi\left(x\frac{d}{dx},\ y\frac{d}{dy},\ z\frac{d}{dz},\ \&c.\right)Ax^my^nz^r\ldots=\phi(m,n,r,\&c.)\,Ax^my^nz^r\ldots;$$

and therefore

$$u=\Sigma\frac{Ax^my^nz^r\ldots}{(am+bn+cr+\&c.\ldots+\alpha)\,(a'm+b'n+c'r+\&c.\ldots+\alpha')\ldots}$$

$$+f_{-\alpha}(x^{\frac{1}{a}},\ y^{\frac{1}{b}},\ z^{\frac{1}{c}},\ \&c.)+f_{-\alpha'}(x^{\frac{1}{a'}},\ y^{\frac{1}{b'}},\ z^{\frac{1}{c'}},\ \&c.)+\&c.,$$

$f_{-\alpha}$, $f_{-\alpha'}$, &c., now denoting *arbitrary* functions of the degree $-\alpha$, $-\alpha'$, &c., homogeneous in the variables which they contain.

I shall now illustrate this method by a few particular examples.

Ex. 1. $x\frac{dz}{dx}-y\frac{dz}{dy}=\frac{x^2}{y}$.

This equation is equivalent to

$$\left(x\frac{d}{dx}-y\frac{d}{dy}\right)z=\frac{x^2}{y};$$

therefore $$z=\frac{1}{x\frac{d}{dx}-y\frac{d}{dy}}\frac{x^2}{y}+f_0(x,y^{-1})$$

$$=\frac{1}{2-(-1)}\frac{x^2}{y}+f(x\div y^{-1})$$

$$=\frac{x^2}{3y}+f(x.y).$$

Gregory's *Examples*, p. 361, *2nd edit.*

Ex. 2. $$x^2\frac{d^2}{dx^2}+x\frac{dz}{dx}-y^2\frac{d^2}{dy^2}-y\frac{dz}{dy}=0.$$

This equation may be put into the form

$$\left\{\left(x\frac{d}{dx}\right)^2-\left(y\frac{d}{dy}\right)^2\right\}z=0,$$

or $$\left\{\left(x\frac{d}{dx}+y\frac{d}{dy}\right)\left(x\frac{d}{dx}-y\frac{d}{dy}\right)\right\}z=0;$$

therefore

$$z=\left(x\frac{d}{dx}+y\frac{d}{dy}\right)^{-1}0+\left(x\frac{d}{dx}-y\frac{d}{dy}\right)^{-1}0=f_0(x,y)+F_0(x,y^{-1})$$

$$=f\left(\frac{x}{y}\right)+F(x.y).$$

Gregory's *Examples*, p. 367.

Ex. 3. $$\left(x^2\frac{d^2}{dx^2}-y^2\frac{d^2}{dy^2}\right)z=xy.$$

This equation may be written

$$\left\{x\frac{d}{dx}\left(x\frac{d}{dx}-1\right)-y\frac{d}{dy}\left(y\frac{d}{dy}-1\right)\right\}z=xy,$$

or $$\left(x\frac{d}{dx}-y\frac{d}{dy}\right)\left(x\frac{d}{dx}+y\frac{d}{dy}-1\right)z=xy;$$

therefore

$$z=\frac{1}{\left(x\frac{d}{dx}-y\frac{d}{dy}\right)\left(x\frac{d}{dx}+y\frac{d}{dy}-1\right)}xy+\left(x\frac{d}{dx}-y\frac{d}{dy}\right)^{-1}0$$

$$+\left(x\frac{d}{dx}+y\frac{d}{dy}-1\right)^{-1}0$$

$$=\frac{1}{\left(x\frac{d}{dx}-y\frac{d}{dy}\right)}xy+f(x.y)+F(x,y)$$

But, as xy is a homogeneous function of the degree zero in x and y^{-1}, we have, by the third of equations (2),

$$z=\tfrac{1}{2}(\log x+\log y^{-1})\,xy+f(xy)+xF\left(\frac{y}{x}\right),$$

and therefore

$$z = \frac{1}{2}xy\{\log x - \log y\} + f(x.y) + xF\left(\frac{y}{x}\right)$$

$$= \frac{1}{2}xy\left(\log\frac{x}{y}\right) + f(x.y) + xF\left(\frac{y}{x}\right).$$

Gregory's *Examples*, p. 365.

This solution is apparently different from Gregory's, being more symmetrical; either, however, may be reduced to the other by a change in $f(x.y)$, which, f being arbitrary, is legitimate.

Ex. 4. $$y^3\frac{dz}{dy} - xy^2\frac{dz}{dx} = axz.$$

Assume $\frac{dz}{z} = du$, or $\log z = u$, which will reduce the equation to $$\left(y^3\frac{d}{dy} - xy^2\frac{d}{dx}\right)u = ax;$$

or $$\left(y\frac{d}{dy} - x\frac{d}{dx}\right)u = \frac{ax}{y^2};$$

hence $$u = \left(y\frac{d}{dy} - x\frac{d}{dx}\right)^{-1}\frac{ax}{y^2} + \left(y\frac{d}{dy} - x\frac{d}{dx}\right)^{-1}0;$$

or $$u = (-2-1)^{-1}\frac{ax}{y^2} + f(x.y);$$

and therefore $$\log z = -\frac{ax}{3y^3} + f(x.y).$$

Gregory's *Examples*, p. 376.

Ex. 5. $x^2r + 2axys + a^2y^2t + px + a^2qy - m^2z = 0.$

The symbolic form of this equation is

$$\left\{\left(x\frac{d}{dx}\right)^2 + 2a\left(x\frac{d}{dx}\right)\left(y\frac{d}{dy}\right) + a^2\left(y\frac{d}{dy}\right)^2 - m^2\right\}z = 0;$$

or $$\left\{\left(x\frac{d}{dx} + ay\frac{d}{dy}\right)^2 - m^2\right\}z = 0;$$

therefore $$z = \left(x\frac{d}{dx} + ay\frac{d}{dy} + m\right)^{-1}0 + \left(x\frac{d}{dx} + ay\frac{d}{dy} - m\right)^{-1}0$$

$$z = f_{-m}(x, y^{\frac{1}{a}}) + F_m(x, y^{\frac{1}{a}})$$

$$z = x^{-m}f\left(\frac{x}{y^{\frac{1}{a}}}\right) + x^mF\left(\frac{x}{y^{\frac{1}{a}}}\right);$$

or $$z = x^{-m}\phi\left(\frac{x^a}{y}\right) + x^m\psi\left(\frac{x^a}{y}\right).$$

If $a = -1$, we get as the solution of

$$x^2r - 2xys + y^2t + px + qy - m^2z = 0,$$

$$z = x^{-m}\phi(x.y) + x^m\psi(x.y).$$

If $a = 1$, we obtain the solution of

$$x^2r + 2xys + y^2t + px + qy - m^2z = 0,$$

$$z = x^{-m}\phi\left(\frac{y}{x}\right) + x^m\psi\left(\frac{y}{x}\right).$$

If $m = 0$, the equation becomes, when reduced to its symbolic form,

$$\left(x\frac{d}{dx} + ay\frac{d}{dy}\right)^2 z = 0;$$

therefore

$$z = \left(x\frac{d}{dx} + ay\frac{d}{dy}\right)^{-2} 0 = f_0(x, y^{\frac{1}{a}}) + (\log x + \log y^{\frac{1}{a}})\, F_0(x, y^{\frac{1}{a}})$$

$$= f\left(\frac{x}{y^{\frac{1}{a}}}\right) + \log(x.y^{\frac{1}{a}})\, F\left(\frac{x}{y^{\frac{1}{a}}}\right)$$

$$= \phi\left(\frac{x^a}{y}\right) + \log(x^a.y)\,\psi\left(\frac{x^a}{y}\right).$$

Ex. 6. $x^2r - y^2t - 2ap + 2bq + (a.a + 1 - b.b + 1)z = 0.$

This equation is equivalent to

$$\left\{\left(x\frac{d}{dx}\right)^2 - (2a+1)x\frac{d}{dx} - \left(y\frac{d}{dy}\right)^2 + (2b+1)y\frac{d}{dy} + a.a + 1 - b.b + 1\right\} z = 0,$$

or $$\left\{x\frac{d}{dx} + y\frac{d}{dy} - (a+b+1)\right\}\left\{x\frac{d}{dx} - y\frac{d}{dy} - (a-b)\right\} z = 0;$$

therefore

$$z = \left\{x\frac{d}{dx} + y\frac{d}{dy} - (a+b+1)\right\}^{-1} 0 + \left\{x\frac{d}{dx} - y\frac{d}{dy} - (a-b)\right\}^{-1} 0$$

$$= f_{a+b+1}(x, y) + F_{a-b}(x, y^{-1})$$

$$= x^{a+b+1} f\left(\frac{x}{y}\right) + x^{a-b} F(x.y).$$

Ex. 7. $$xy\frac{d^2z}{dx\,dy}+ax\frac{dz}{dx}+by\frac{dz}{dy}+abz=V.$$

This equation may be written in the form

$$\left(x\frac{d}{dx}+b\right)\left(y\frac{d}{dy}+a\right)z=V;$$

therefore

$$z=\frac{1}{\left(x\frac{d}{dx}+b\right)\left(y\frac{d}{dy}+a\right)}V+\left(x\frac{d}{dx}+b\right)^{-1}0+\left(y\frac{d}{dy}+a\right)^{-1}0$$

$$=\frac{1}{\left(x\frac{d}{dx}+b\right)\left(y\frac{d}{dy}+a\right)}V+x^{-b}f(y)+y^{-a}F(x).$$

Gregory's *Examples*, p. 366.

When the form of V has been assigned, the term $\frac{1}{\left(x\frac{d}{dx}+b\right)\left(y\frac{d}{dy}+a\right)}V$ can be interpreted. In fact, if $V=\Sigma Ax^my^n$, it is equivalent to $\Sigma\frac{Ax^my^n}{(m+b)(n+a)}$.

By this method can be integrated all equations of the form

$$\left\{\left(x\frac{d}{dx}+b\frac{d}{dy}+\&c.+\alpha\right)\left(x\frac{d}{dx}+b'\frac{d}{dy}+\&c.+\alpha'\right)\cdots\right\}u=V,$$

where the operator affecting u, so far as it depends on any one of the variables, z for example, may contain either $\frac{d}{dz}$ or $z\frac{d}{dz}$, but not both. For if we put $y=\log y_1$, $\frac{d}{dy}=y_1\frac{d}{dy_1}$, and the equation is reduced to the form

$$\left\{\left(x\frac{d}{dx}+by_1\frac{d}{dy_1}+\&c.+\alpha\right)\left(x\frac{d}{dx}+b'y_1\frac{d}{dy_1}+\&c.+\alpha'\right)\cdots\right\}u=V,$$

which is comprehended under the class of equations already considered.

This being solved, we must change y_1 into e^y in the result.

As an example of this class, suppose

$$\left(x^2\frac{d^2}{dx^2}+x\frac{d}{dx}+ax\frac{d^2}{dx\,dy}+b\frac{d^2}{dy^2}\right)z=xy,$$

or

$$\left\{\left(x\frac{d}{dx}\right)^2+ax\frac{d}{dx}\frac{d}{dy}+b\left(\frac{d}{dy}\right)^2\right\}z=xy,$$

or $$\left(x\frac{d}{dx}-\alpha\frac{d}{dy}\right)\left(x\frac{d}{dx}-\beta\frac{d}{dy}\right)z=xy,$$

α, β being the roots of the equation

$$u^2+au+b=0;$$

therefore

$$z=\frac{1}{\left(x\frac{d}{dx}-\alpha\frac{d}{dy}\right)\left(x\frac{d}{dx}-\beta\frac{d}{dy}\right)}xy$$
$$+\left(x\frac{d}{dx}-\alpha\frac{d}{dy}\right)^{-1}0+\left(x\frac{d}{dx}-\beta\frac{d}{dy}\right)^{-1}0,$$

but $$\left(x\frac{d}{dx}-\alpha\frac{d}{dy}\right)^{-1}0=\left(x\frac{d}{dx}-\alpha y_1\frac{d}{dy_1}\right)^{-1}0=f(x.y_1^{\frac{1}{\alpha}})$$
$$=f(x.e^{\frac{y}{\alpha}})=f(e^{\log x}.e^{\frac{y}{\alpha}})=f(e^{\frac{y}{\alpha}+\log x})$$
$$=\phi\left(\frac{y}{\alpha}+\log x\right),\text{ or, as }\phi\text{ is arbitrary,}$$
$$=\phi(y+\alpha\log x);$$

therefore

$$z=\frac{1}{\left(1-\alpha\frac{d}{dy}\right)\left(1-\beta\frac{d}{dy}\right)}xy+\phi(y+\alpha\log x)+\psi(y+\beta\log x),$$

but $$\frac{1}{\left(1-\alpha\frac{d}{dy}\right)\left(1-\beta\frac{d}{dy}\right)}xy=\frac{x}{\alpha\beta}\left(\frac{d}{dy}-\frac{1}{\alpha}\right)^{-1}\left(\frac{d}{dy}-\frac{1}{\beta}\right)^{-1}y$$
$$=\frac{x}{b}e^{\frac{y}{\alpha}}\int_y e^{-\frac{y}{\alpha}}e^{\frac{y}{\beta}}\int_y e^{-\frac{y}{\beta}}y=x(y-a),$$

and the solution is

$$z=x(y-a)+\phi(y+\alpha\log x)+\psi(y+\beta\log x).$$

II. Many of the most important differential equations of the second order and first degree as yet integrated, indeed all those given by Gregory in his *Examples* which are not reducible either to equations with constant coefficients, or to the class considered in the first part of this paper, are reducible to the following form:

$$\left(\frac{d^2}{dx^2}+2Q\frac{d}{dx}+Q^2+Q'\pm c^2-\frac{m.m+1}{x^2}\right)u=0\ldots(\alpha),$$

Q being a function of x and $Q'=\frac{d}{dx}Q$.

It is therefore a matter of some importance to obtain the solution of this equation in a form easily capable of interpretation in all cases, more particularly so, as in the instances which Gregory has considered the integration has been effected by various means, and frequently by series, a method which is always tedious, and involves the difficulty of reducing the result when obtained to a finite form. Dr. Hargreave has solved the equation

$$\left(\frac{d^2}{dx^2} + 2Q\frac{d}{dx} + Q^2 + Q' - c^2 - \frac{m.m-1}{x^2}\right)u = 0,^*$$

which may be identified with (α) by changing m into $m+1$. His solution then involving $\left(\frac{d^2}{dx^2} - c^2\right)^m$ or $\left(e^{cx}\frac{d}{dx}e^{-2cx}\frac{d}{dx}e^{cx}\right)^m$, in order to interpret it we must perform $2m$ differentiations. But a solution involving only half that number, and therefore more immediately interpretable, may be obtained as follows.

The equation

$$\left(\frac{d^2}{dx^2} + 2Q\frac{d}{dx} + Q^2 + Q' \pm c^2 - \frac{m.m+1}{x^2}\right)u = 0$$

may be put into the form

$$\left\{\left(\frac{d}{dx} + Q\right)^2 \pm c^2 - \frac{m.m+1}{x^2}\right\}u = 0,$$

or $$\left\{\left(e^{-Q_1}\frac{d}{dx}e^{Q_1}\right)^2 \pm c^2 - \frac{m.m+1}{x^2}\right)u = 0, \text{ where } Q_1 = \int Q\,dx,$$

or $$\left\{e^{-Q_1}\left(\frac{d}{dx}\right)^2 e^{Q_1} \pm c^2 - \frac{m.m+1}{x^2}\right\}u = 0;$$

therefore $$\left\{\frac{d^2}{dx^2} \pm c^2 - \frac{m.m+1}{x^2}\right\}e^{Q_1}u = 0.$$

Let $e^{Q_1}u = z$, and therefore $u = e^{-Q_1}z$,

where z is given by the equation

$$\left(\frac{d^2}{dx^2} \pm c^2 - \frac{m.m+1}{x^2}\right)z = 0.$$

This equation, by a very ingenious process, has been solved by Professor Boole in the following elegant form,

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* *Philosophical Transactions* for 1848, Part I.

$$z = \frac{1}{x^{m+1}} \left(x^3 \frac{d}{dx} \right)^m \frac{1}{x^{2m-1}} \left(\frac{d^2}{dx^2} \pm c^2 \right)^{-1} 0,$$

which is equivalent to

$$z = \frac{1}{x^{m+1}} \left(x^3 \frac{d}{dx} \right)^m \frac{1}{x^{2m-1}} \{ A \cos(cx + \alpha) \},$$

or

$$z = \frac{1}{x^{m+1}} \left(x^3 \frac{d}{dx} \right)^m \frac{1}{x^{2m-1}} (Ae^{cx} + Be^{-cx}),$$

according as the upper or lower sign before c^2 has been taken throughout.*

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* Professor Boole's paper being difficult of access to many readers of this Journal, I may be excused for giving here his method of obtaining the solution which I have assumed above. The equation

$$\left(\frac{d^2}{dx^2} \pm c^2 - \frac{m.m+1}{x^2} \right) u = 0,$$

by assuming $x = e^\theta$, can be reduced to the form

$$\{ (D+m)\ (D-m-1) \pm c^2 e^{2\theta} \}\, u = 0,$$

where $D = \frac{d}{d\theta}$. Let $u = e^{-m\theta} v$, and we get

$$\{ D\,(D-2m-1) \pm c^2 e^{2\theta} \}\, v = 0.$$

Assume $v = \phi(D)\, w$, and this equation becomes

$$\{ D(D-2m-1)\ \phi(D) \pm c^2 \phi(D-2)\ e^{2\theta} \}\, w = 0,$$

and therefore

$$\{ D(D-2m-1) \frac{\phi(D)}{\phi(D-2)} \pm c^2 e^{2\theta} \}\, w = \{ \phi(D-2) \}^{-1} 0.$$

This can be assimilated to the equation

$$\{ D(D-1) \pm c^2 e^{2\theta} \}\, w = 0,$$

by assuming $\{\phi(D-2)\}^{-1} 0 = 0$, and determining $\phi(D)$ so as to satisfy the equation

$$D(D-2m-1) \frac{\phi(D)}{\phi(D-2)} = D.D-1, \quad \text{or} \quad \frac{\phi(D)}{\phi(D-2)} = \frac{D-1}{D-2m-1},$$

which may be done by assuming

$$\phi(D) = \frac{D-1.D-3.D-5 \ldots}{D-2m-1.D-2m-3 \ldots} = (D-1.D-3.D-5 \ldots D-2m+1)$$

$$= e^\theta\, De^{-\theta}\, e^{3\theta}\, De^{-3\theta} \ldots De^{(-2m+1)\theta} = e^{-\theta}\, (e^{3\theta}\, D)^m\, e^{(-2m+1)\theta}$$

$$= \frac{1}{x} \left(x^3 \frac{d}{dx} \right)^m \frac{1}{x^{2m-1}}, \quad \text{since } D = x \frac{d}{dx},$$

therefore

$$u = x^{-m} v = x^{-m}\, \phi(D)\, w = \frac{1}{x^{m+1}} \left(x^3 \frac{d}{dx} \right)^m \frac{1}{x^{2m-1}}\, w.$$

But the equation for w is equivalent to

$$\left\{ x \frac{d}{dx} \left(x \frac{d}{dx} - 1 \right) \pm c^2 x^2 \right\} w = 0,$$

The corresponding solutions of (α) are

$$u = \frac{e^{-Q_1}}{x^{m+1}}\left(x^3\frac{d}{dx}\right)^m \frac{A\cos(cx+\alpha)}{x^{2m-1}},$$

and $$u = \frac{e^{-Q_1}}{x^{m+1}}\left(x^3\frac{d}{dx}\right)^m \frac{Ae^{cx}+Be^{-cx}}{x^{2m-1}}.$$

Suppose, for example, $Q = \frac{n}{x}$, and we get the solution of the equation

$$\left(D^2 + \frac{2n}{x}D + \frac{n.n-1-m.m+1}{x^2} \pm c^2\right)u = 0, \text{ where } D = \frac{d}{dx},$$

in the form

$$u = \frac{1}{x^{m+n+1}}\left(x^3\frac{d}{dx}\right)^m \frac{A\cos(cx+\alpha)}{x^{2m-1}},$$

or $$u = \frac{1}{x^{m+n+1}}\left(x^3\frac{d}{dx}\right)^m \frac{Ae^{cx}+Be^{-cx}}{x^{2m-1}}.$$

If $m = 1$, $n = 1$, we obtain as the solution of

$$\left(D^2 + \frac{2}{x}D + c^2 - \frac{2}{x^2}\right)u = 0,$$

$$u = \frac{d}{dx}\left\{\frac{A\cos(cx+\alpha)}{x}\right\} = -\frac{A}{x^2}\cos(cx+\alpha) - \frac{Ac}{x}\sin(cx+\alpha),$$

Gregory's *Examples*, p. 313;

while the solution of

$$\left(D^2 + \frac{2}{x}D - c^2 - \frac{2}{x^2}\right)u = 0,$$

is $$u = \frac{d}{dx}\left\{\frac{Ae^{cx}+Be^{-cx}}{x}\right\} = A\left(\frac{c}{x} - \frac{1}{x^2}\right)e^{cx} - B\left(\frac{c}{x} + \frac{1}{x^2}\right)e^{-cx}.$$

The suppositions $n = 1$, $m = 0$, give the equation

$$\left(D^2 + \frac{2}{x}D \pm c^2\right)u = 0,$$

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or $$\left(x^3\frac{d^2}{dx^2} \pm c^2x^3\right)w = 0, \quad \text{or} \quad \left(\frac{d^2}{dx^2} \pm c^2\right)w = 0;$$

therefore $$w = \left(\frac{d^2}{dx^2} \pm c^2\right)^{-1}0;$$

and therefore $$u = \frac{1}{x^{m+1}}\left(x^3\frac{d}{dx}\right)^m \frac{1}{x^{2m-1}}\left(\frac{d^2}{dx^2} \pm c^2\right)^{-1}0.$$

and its solution is

$$u = \frac{1}{x}\{A\cos(cx+\alpha)\},$$

or

$$u = \frac{1}{x}\{Ae^{cx}+Be^{-cx}\},$$

according as we take the upper or lower sign before c^2 in the proposed equation.

Gregory's *Examples*, p. 312.

Let $Q = c = \frac{1}{2}q$, and take the negative sign before c^2; the equation is then

$$\left(D^2 + qD - \frac{m.m+1}{x^2}\right)u = 0,$$

and its solution

$$u = \frac{e^{-\frac{1}{2}qx}}{x^{m+1}}\left(x^3\frac{d}{dx}\right)^m \frac{Ae^{\frac{1}{2}qx}+Be^{-\frac{1}{2}qx}}{x^{2m-1}}.$$

If, for instance, $m = 1$, we obtain the equation

$$\left(D^2 + qD - \frac{2}{x^2}\right)u = 0,$$

and its solution

$$u = e^{-\frac{1}{2}qx}\,x\,\frac{d}{dx}\,\frac{Ae^{\frac{1}{2}qx}+Be^{-\frac{1}{2}qx}}{x}$$

$$= C_1\left(1-\frac{2}{qx}\right)+C_2\left(1+\frac{2}{qx}\right)e^{-qx}.$$

Gregory's *Examples*, p. 349.

Let $Q = \frac{-2}{x}$, $m = 2$, and take the positive sign before c^2, we thus get the equation

$$\left(D^2 - \frac{4}{x}D + c^2\right)u = 0,$$

and its solution

$$u = \frac{1}{x}\left(x^3\frac{d}{dx}\right)^2\frac{A\cos(cx+\alpha)}{x^3}$$

$$= x^2\frac{d}{dx}\,x^3\frac{d}{dx}\,\frac{A\cos(cx+\alpha)}{x^3}$$

$$= A\{(3-c^2x^2)\cos(cx+\alpha)+3cx\sin(cx+\alpha)\}.$$

Gregory's *Examples*, p. 350.

If in equation (α) we put $Q = \frac{n}{x}$, and transform, as Dr. Hargreave has done in the paper already alluded to, by changing the independent variable so as to cause the second term to vanish, we get the equation

$$\left\{\frac{d^2}{dz^2} \pm \left(\frac{c}{2n-1}\right)^2 z^{-\frac{4n}{2n-1}} + (n.n-1-m.m+1)\, z^{-2}\right\} u = 0, \text{ where } x = z^{-\frac{1}{2n-1}},$$

which for $m = n - 1$ reduces itself to

$$\left\{\frac{d^2}{dz^2} \pm \left(\frac{c}{2n-1}\right)^2 z^{-\frac{4n}{2n-1}}\right\} u = 0 \ldots\ldots\ldots\ldots (\beta);$$

the solution of which therefore is

$$u = x^{-2n}\left(x^3 \frac{d}{dx}\right)^{n-1} \frac{1}{x^{2n-3}} \left(\frac{d^2}{dx^2} \pm c^2\right)^{-1} 0,$$

which is equivalent to

$$\left.\begin{aligned} u &= x^{-2n}\left(x^3 \frac{d}{dx}\right)^{n-1} \frac{A\cos(cx+\alpha)}{x^{2n-3}} \\ \text{or} \quad u &= x^{-2n}\left(x^3 \frac{d}{dx}\right)^{n-1} \frac{Ae^{cx} + Be^{-cx}}{x^{2n-3}} \end{aligned}\right\}, \text{ where } x = z^{-\frac{1}{2n-1}},$$

according as the upper or lower sign before c^2 is taken in (β).

If in equation (α) we put $Q = \frac{-n}{x}$, and change the independent variable so as to cause the second term to vanish, or change, in the transformed equation above, n into $-n$, we obtain the equation

$$\left\{\frac{d^2}{dz^2} \pm \left(\frac{c}{2n+1}\right)^2 z^{-\frac{4n}{2n+1}} + (n.n+1-m.m+1) z^{-2}\right\} u = 0, \text{ where } x = z^{\frac{1}{2n+1}}.$$

If we assume $m = n$, this equation reduces itself to

$$\left\{\frac{d^2}{dz^2} \pm \left(\frac{c}{2n+1}\right)^2 z^{-\frac{4n}{2n+1}}\right\} u = 0 \ldots\ldots\ldots\ldots (\gamma),$$

the solution of which is

$$u = x^{-1}\left(x^3 \frac{d}{dx}\right)^{n} \frac{1}{x^{2n-1}} \left(\frac{d^2}{dx^2} \pm c^2\right)^{-1} 0,$$

which is equivalent to

$$\left.\begin{aligned} u &= x^{-1}\left(x^3\frac{d}{dx}\right)^n \frac{A\cos(cx+\alpha)}{x^{2n-1}} \\ \text{or}\quad u &= x^{-1}\left(x^3\frac{d}{dx}\right)^n \frac{Ae^{cx}+Be^{-cx}}{x^{2n-1}} \end{aligned}\right\}, \text{ where } x = z^{\frac{1}{2n+1}}.$$

according as the upper or lower sign is taken in the second term of (γ).

If in (β) and (γ) we substitute $e^{\int z^v}$ for u, they will be reduced to Riccati's forms; but the solutions are not interpretable unless n be an integer, which is Riccati's restriction.

Substituting unity for (n) in (β), we get the equation

$$\left(\frac{d^2}{dz^2} - \frac{c^2}{z^4}\right)u = 0,$$

and its solution

$$u = x^{-1}\frac{(Ae^{cx}+Be^{-cx})}{x^{-1}}, \text{ where } x = \frac{1}{z},$$

or $$u = z\left(Ae^{\frac{c}{z}} + Be^{-\frac{c}{z}}\right);$$

while the solution of $\left(\frac{d^2}{dz^2} + \frac{c^2}{z^4}\right)u = 0$ is $u = Az\cos\left(\frac{c}{z}+\alpha\right)$.

Gregory's *Examples*, p. 345.

In (γ) let $n = 1$, and taking the negative sign before c^2 it becomes

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{3}\right)^2 z^{-\frac{4}{3}}\right\}u = 0;$$

the solution of which is

$$\begin{aligned} u &= x^{-1}\left(x^3\frac{d}{dx}\right)\left(\frac{Ae^{cx}+Be^{-cx}}{x}\right), \text{ where } x = z^{\frac{1}{3}}, \\ &= A(cx-1)e^{cx} - B(cx+1)e^{-cx} \\ &= A(cz^{\frac{1}{3}}-1)e^{cz^{\frac{1}{3}}} - B(cz^{\frac{1}{3}}+1)e^{-cz^{\frac{1}{3}}}; \end{aligned}$$

or supposing $c = 3k$, the solution of $\left(\frac{d^2}{dz^2} - k^2z^{-\frac{4}{3}}\right)u = 0$ is

$$u = C_1\left(z^{\frac{1}{3}} - \frac{1}{3k}\right)e^{3kz^{\frac{1}{3}}} + C_2\left(z^{\frac{1}{3}} + \frac{1}{3k}\right)e^{-3kz^{\frac{1}{3}}}.$$

Gregory's *Examples*, p. 345.

In the same manner the solution of $\left(\frac{d^2}{dz^2} + k^2 z^{-\frac{4}{3}}\right) u = 0$ is

$$u = C\left\{z^{\frac{1}{3}} \sin(3kz^{\frac{1}{3}} + \alpha) + \frac{1}{3k} \cos(3kz^{\frac{1}{3}} + \alpha)\right\}.$$

In (γ) let $n = 2$; and taking the negative sign before c^2 we get the equation

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{5}\right)^2 z^{-\frac{8}{5}}\right\} u = 0,$$

the solution being

$$u = x^{-1}\left(x^3 \frac{d}{dx}\right)^2 \frac{Ae^{cx} + Be^{-cx}}{x^3}, \quad \text{where } x = z^{\frac{1}{5}},$$

or $$u = A(c^2x^2 - 3cx + 3)\, e^{cx} + B(c^2x^2 + 3cx + 3)\, e^{-cx},$$

or $$u = A(c^2z^{\frac{2}{5}} - 3cz^{\frac{1}{5}} + 3)\, e^{cz^{\frac{1}{5}}} + B(c^2z^{\frac{2}{5}} + 3cz^{\frac{1}{5}} + 3)\, e^{-cz^{\frac{1}{5}}};$$

and therefore the solution of $\left(\frac{d^2}{dz^2} - k^2 z^{-\frac{8}{5}}\right) u = 0$ is

$$u = C_1\left(z^{\frac{2}{5}} - \frac{3}{5k} z^{\frac{1}{5}} + \frac{3}{(5k)^2}\right) e^{5kz^{\frac{1}{5}}} + C_2\left(z^{\frac{2}{5}} + \frac{3}{5k} z^{\frac{1}{5}} + \frac{3}{(5k)^2}\right) e^{-5kz^{\frac{1}{5}}}.$$

This equation is given by Gregory (p. 345), but his solution is incorrect, as is evident from inspection, since the particular solutions should only differ in the sign of k.

We have hitherto in equation (α) considered Q to be a function of x only, and c to be a constant, in which case (α) will represent a class of linear differential equations. But if we suppose Q a function of x and $\frac{d}{dy}$, and c a function of $\frac{d}{dy}$, (α) will then represent a class of partial differential equations, the most general form of which will be

$$\left[\frac{d^2}{dx^2} + 2f\left(x, \frac{d}{dy}\right)\frac{d}{dx} + \left\{f\left(x, \frac{d}{dy}\right)\right\}^2 + f'\left(x, \frac{d}{dy}\right) - \frac{m.m+1}{x^2} - \left\{F\left(\frac{d}{dy}\right)\right\}^2\right] u = 0 \ldots(\alpha'),$$

where $f'\left(x, \frac{d}{dy}\right)$ stands for $\frac{d}{dx} f\left(x, \frac{d}{dy}\right)$, the corresponding solution being

$$u = \frac{e^{-\int f\left(x, \frac{d}{dy}\right) dx}}{x^{m+1}} \left(x^3 \frac{d}{dx}\right)^m \frac{1}{x^{2m-1}} \left[\frac{d^2}{dx^2} - \left\{F\left(\frac{d}{dy}\right)\right\}^2\right]^{-1} 0.$$

If $F\left(\frac{d}{dy}\right) = c\frac{d}{dy}$, this solution becomes

$$u = \frac{e^{-\int f\left(x, \frac{d}{dy}\right) dx}}{x^{m+1}} \left(x^3 \frac{d}{dx}\right)^m \frac{1}{x^{2m-1}} \left\{e^{cx\frac{d}{dy}}\phi(y) + e^{-cx\frac{d}{dy}}\psi(y)\right\},$$

or $$u = \frac{e^{-\int f\left(x, \frac{d}{dy}\right) dx}}{x^{m+1}} \left(x^3 \frac{d}{dx}\right)^m \frac{1}{x^{2m-1}} \left\{\phi(y+cx) + \psi(y-cx)\right\}.$$

Suppose, for example,

$$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - a^2\frac{d^2}{dy^2}\right) u = 0.$$

In this case $f\left(x, \frac{d}{dy}\right) = \frac{1}{x}$, $m = 0$, and the solution is

$$u = \frac{1}{x}\left\{\phi(y+ax) + \psi(y-ax)\right\}.$$

Gregory's *Examples*, p. 367.

Let $$\frac{d^2}{dy^2}u = a^2\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{2}{x^2}\right)u,$$

or $$\left(\frac{d^2}{dx^2} + \frac{2}{x}\frac{d}{dx} - \frac{2}{x^2} - \frac{1}{a^2}\frac{d^2}{dy^2}\right) u = 0.$$

Here $f\left(x, \frac{d}{dy}\right) = \frac{1}{x}$, $m = 1$, $c = \frac{1}{a}$, and the solution consequently is

$$u = \frac{d}{dx}\frac{1}{x}\left\{\phi\left(y + \frac{x}{a}\right) + \psi\left(y - \frac{x}{a}\right)\right\},$$

or, as ϕ and ψ are arbitrary,

$$u = \frac{d}{dx}\frac{1}{x}\left\{\phi(x+ay) + \psi(x-ay)\right\}$$

$$= \frac{1}{x}\left\{\phi'(x+ay) + \psi'(x-ay)\right\} - \frac{1}{x^2}\left\{\phi(x+ay) + \psi(x-ay)\right\}.$$

Gregory's *Examples*, p. 368.

Let $$\left(\frac{d^2}{dx^2} - \frac{2}{x^2} - a^2\frac{d^2}{dy^2}\right)u = 0.$$

Here $f\left(x, \frac{d}{dy}\right) = 0$, $m = 1$, and the solution is

$$u = x\frac{d}{dx}\frac{1}{x}\{\phi(y+ax)+\psi(y-ax)\}$$
$$= a\phi'(y+ax) - a\psi'(y-ax) - \frac{1}{x}\{\phi(y+ax)+\psi(y-ax)\}.$$

Gregory's *Examples*, p. 368.

Let $$\left(\frac{d^2}{dx^2} + a\frac{d^2}{dx\,dy} - \frac{2}{x^2}\right)u = 0.$$

Here $f\left(x, \frac{d}{dy}\right) = \frac{a}{2}\frac{d}{dy}$, $c = \frac{a}{2}$, and $m = 1$; therefore

$$u = e^{-\frac{a}{2}x\frac{d}{dy}}x\frac{d}{dx}\frac{1}{x}\left\{\phi\left(y+\frac{ax}{2}\right)+\psi\left(y-\frac{ax}{2}\right)\right\}$$
$$= e^{-\frac{a}{2}x\frac{d}{dy}}\left[\frac{a}{2}\left\{\phi'\left(y+\frac{ax}{2}\right)-\psi'\left(y-\frac{ax}{2}\right)\right\} - \frac{1}{x}\left\{\phi\left(y+\frac{ax}{2}\right)+\psi\left(y-\frac{ax}{2}\right)\right\}\right]$$
$$= \frac{a}{2}\phi'(y) - \frac{1}{x}\phi(y) - \frac{a}{2}\psi'(y-ax) - \frac{1}{x}\psi(y-ax);$$

or, as ϕ and ψ are arbitrary,

$$u = a\phi'(y) - \frac{2}{x}\phi(y) + a\psi'(y-ax) + \frac{2}{x}\psi(y-ax).$$

Gregory's *Examples*, p. 369.

Let $$\left(\frac{d^2}{dx^2} - 2\frac{p}{x}\frac{d}{dx} - a^2\frac{d^2}{dy^2}\right)u = 0.$$

Here $f\left(x, \frac{d}{dy}\right) = -\frac{p}{x}$ and $m = p$; the solution therefore is

$$u = \frac{1}{x}\left(x^3\frac{d}{dx}\right)^p \frac{\phi(y+cx)+\psi(y-cx)}{x^{2p-1}}.$$

If $p = 2$, the equation becomes

$$\left(\frac{d^2}{dx^2} - \frac{4}{x}\frac{d}{dx} - a^2\frac{d^2}{dy^2}\right)u = 0,$$

the solution of which therefore is

$$u = x^2 \frac{d}{dx} x^3 \frac{d}{dx} \frac{\phi(y+ax) + \psi(y-ax)}{x^3}$$

$$= x^2 \frac{d}{dx}\left[\frac{-3}{x}\{\phi(y+ax)+\psi(y-ax)\}+a\{\phi'(y+ax)-\psi'(y-ax)\}\right]$$

$$= 3\{\phi(y+ax)+\psi(y-ax)\} - 3ax\{\phi'(y+ax)-\psi'(y-ax)\}$$
$$+ a^2x^2\{\phi''(y+ax)+\psi''(y-ax)\}.$$

Gregory's *Examples*, p. 369.

If in (β) and (γ) we change c into $c\frac{d}{dy}$, we obtain the equations

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{2n-1}\right)^2 z^{-\frac{4n}{2n-1}} \frac{d^2}{dy^2}\right\} u = 0 \ldots\ldots\ldots (\beta'),$$

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{2n+1}\right)^2 z^{-\frac{4n}{2n+1}} \frac{d^2}{dy^2}\right\} u = 0 \ldots\ldots\ldots (\gamma');$$

the solutions of which therefore are respectively

$$u = x^{-2n}\left(x^3\frac{d}{dx}\right)^{n-1} \frac{\phi(y+cx)+\psi(y-cx)}{x^{2n-3}}, \text{ where } x = z^{-\frac{1}{2n-1}},$$

$$u = x^{-1}\left(x^3\frac{d}{dx}\right)^{n} \frac{\phi(y+cx)+\psi(y-cx)}{x^{2n-1}}, \quad \text{where } x = z^{\frac{1}{2n+1}}.$$

In (β') let $n = 1$; it will then become

$$\left(\frac{d^2}{dz^2} - \frac{c^2}{z^4}\frac{d^2}{dy^2}\right) u = 0,$$

the solution of which therefore is

$$u = x^{-1}\{\phi(y+cx)+\psi(y-cx)\}, \text{ where } x = z^{-1},$$

or
$$u = z\left\{\phi\left(y+\frac{c}{z}\right)+\psi\left(y-\frac{c}{z}\right)\right\}.$$

Gregory's *Examples*, p. 368.

In (γ') let $n = 1$, and we obtain the equation

$$\left\{\frac{d^2}{dz^2} - \left(\frac{c}{3}\right)^2 z^{-\frac{4}{3}}\frac{d^2}{dy^2}\right\} u = 0,$$

the solution of which consequently is

$$u = x^2 \frac{d}{dx} \frac{\phi(y+cx) + \psi(y-cx)}{x}, \text{ where } x = z^{\frac{1}{3}}$$

$$= -\{\phi(y+cx) + \psi(y-cx)\} + cx\{\phi'(y+cx) - \psi'(y-cx)\}$$

$$= -\{\phi(y+cz^{\frac{1}{3}}) + \psi(y-cz^{\frac{1}{3}})\} + cz^{\frac{1}{3}}\{\phi'(y+cz^{\frac{1}{3}}) - \psi'(y-cz^{\frac{1}{3}})\}.$$

The solution of $\left(\frac{d^2}{dz^2} - a^2 z^{-\frac{4}{3}} \frac{d^2}{dy^2}\right) u = 0$ consequently is

$$u = -\{\phi(y+3az^{\frac{1}{3}}) + \psi(y-3az^{\frac{1}{3}})\} + 3az^{\frac{1}{3}}\{\phi'(y+3az^{\frac{1}{3}}) - \psi'(y-3az^{\frac{1}{3}})\};$$

or, as ϕ and ψ are arbitrary,

$$u = z^{\frac{1}{3}}\{\phi'(y+3az^{\frac{1}{3}}) + \psi'(y-3az^{\frac{1}{3}})\} - \frac{1}{3a}\{\phi(y+3az^{\frac{1}{3}}) - \psi(y-3az^{\frac{1}{3}})\}.$$

Gregory's *Examples*, p. 368.

6, *Trinity College, Dublin, Nov.* 24, 1854.

THE END.

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